SUBIDEAL BORDER BASES

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Abstract. In modeling physical systems, it is sometimes useful to construct border bases of 0-dimensional polynomial ideals which are contained in the ideal generated by a given set of polynomials. We define and construct such subideal border bases, provide some basic properties and generalize a suitable variant of the Buchberger-Möller algorithm as well as the AVI-algorithm of [5] to the subideal setting. The subideal version of the AVI-algorithm is then applied to an actual industrial problem.

Contents

1. Introduction

In [5] an algorithm was introduced which computes an approximate border basis consisting of unitary polynomials that vanish approximately at a given set of points. It has been shown that this *AVI-algorithm* is useful for modeling physical systems based on a set of measured data points. More precisely, given a finite point set $\mathbb{X} = \{p_1, \dots, p_s\} \subset [-1,1]^n$, the AVI-algorithm computes an order ideal $\mathcal{O}$ of terms in $\mathbb{T}^n$ and an $\mathcal{O}$-border prebasis $G = \{g_1, \dots, g_\nu\}$ such that

(1) the unitary polynomials $g_i/\|g_i\|$ vanish ε-approximately at $\mathbb{X}$, where $\varepsilon > 0$ is a given threshold number, and
(2) the normal remainders of the S-polynomials $S(g_i, g_j)$ for g_i, g_j with neighboring border terms are smaller than ε.

Abstractly speaking, the last condition means that the point in the moduli space corresponding to G is "close" to the border basis scheme (see [11] and [8]). In practical applications, the AVI-algorithm turns out to be very stable and useful. With a judicial choice of the threshold number ε, it is able to discover simple polynomial relations which exist in the data with high reliability. For instance, it discovers simple physical laws inherent in measured data without the need of imposing model equations.

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However, in some situations physical information may be available which is not contained in the data points $\mathbb{X}$ or we may have exact physical knowledge which is only approximately represented by the data points. An example for this phenomenon will be discussed in Section 6. For instance, we may want to *impose* certain vanishing conditions on the model equations we are constructing. Using Hilbert's Nullstellensatz this translates to saying that what we are looking for is the intersection of the vanishing ideal of $\mathbb{X}$ with a given ideal $J \subseteq \mathbb{R}[x_1, \dots, x_n]$ whose generators represent the vanishing conditions we want to impose.

In order to be able to deal with this approximate situation, it is first necessary to generalize the exact version of the computation of vanishing ideals to the subideal setting. Then this theory will serve as a guide and a motivation for the approximate case. Therefore this paper begins in Section 2 with the definition and basic properties of subideal border bases.

Given a 0-dimensional ideal I in a polynomial ring $P = K[x_1, \dots, x_n]$ over a field and a set of polynomials $F = \{f_1, \dots, f_m\}$ generating an ideal $J = \langle F \rangle$, a subideal border basis of I corresponds to a set of polynomials $\mathcal{O}_F = \mathcal{O}_1 \cdot f_1 \cup \cdots \cup \mathcal{O}_m \cdot f_m$, where the $\mathcal{O}_i$ are order ideals of terms, such that the residue classes of the elements of $\mathcal{O}_F$ form a K-basis of $J/(I \cap J) \cong (I + J)/I$. Clearly, this generalizes the case $F = \{1\}$, i.e. the "usual" border basis theory. We show that subideal border bases always exist and explain a method to construct them from a border basis of I. Moreover, we discuss some uniqueness properties of subideal border bases.

The foundation of any further development of the theory of subideal border bases is a generalization of the Border Division Algorithm (see [10], 6.4.11) to the subideal case. This foundation is laid in Section 3 where we also study higher $\mathcal{O}_F$-borders, the $\mathcal{O}_F$-index, and show that a subideal border basis of I generates $I \cap J$.

In Section 4, we generalize the Buchberger-Möller algorithm (BM-algorithm) for computing vanishing ideals of point sets to the subideal setting. More precisely, we generalize a version of the BM-algorithm which proceeds blockwise degree-by-degree and produces a border basis of the vanishing ideal. Similarly, the subideal version of the BM-algorithm (cf. Algorithm 4.2) computes an $\mathcal{O}_\sigma(I_\mathbb{X})_F$-subideal border basis of $I_\mathbb{X}$, where $\mathcal{O}_\sigma(I_\mathbb{X})$ is the complement of a leading term ideal of the vanishing ideal $I_\mathbb{X}$ of $\mathbb{X}$.

Next, in Section 5, we turn to the setting of Approximate Computational Algebra. We work in the polynomial ring $\mathbb{R}[x_1, \dots, x_n]$ over the reals and assume that $\mathbb{X} \subset [-1, 1]^n$ is a finite set of (measured, imprecise) points. We define approximate $\mathcal{O}_F$-subideal border bases and generalize the AVI-algorithm from [5], Thm. 3.3 to the subideal case.

Let us point out that the subideal version of the AVI algorithm contains a substantial difference to the traditional way of computing approximate vanishing ideals, e.g. as in [1]. Namely, the AVI algorithm produces a set of polynomials which vanish approximately at the given data points, but we do not demand that there exists a "nearby" set of points at which these polynomials vanish exactly. The latter requirement has turned out to be too restrictive for real-world applications, for instance the one we explain in the last section. There we provide an example for the application of these techniques to the problem of production allocation in the oil industry.

Unless explicitly stated otherwise, we use the notation and definitions of [9] and [10]. We shall assume that the reader has some familiarity with the theory of exact and approximate border bases (see for instance [5], [6], [7], [8], Section 6.4 of [10], and [12]).

2. Subideal Border Bases

Here we are interested in a "relative" version of the notion of border bases in the following sense. Let K be a field, let $P = K[x_1, \dots, x_n]$ be a polynomial ring, let $\mathbb{T}^n$ be its monoid of terms, let $\mathcal{O}$ be an order ideal in $\mathbb{T}^n$, and let $I \subset P$ be a 0-dimensional ideal.

Suppose we are given a further polynomial ideal $J = \langle f_1, \dots, f_m \rangle$ of P, where $F = \{f_1, \dots, f_m\} \subset P \setminus \{0\}$. Our goal is to describe and compute the intersection ideal $I \cap J$ as a subideal of J. By Noether's isomorphism theorem, we have $J/(I \cap J) \cong (I + J)/I \subset P/I$. Therefore J has a finite K-vector space basis modulo $I \cap J$. Now we are looking for the following special kind of vector space basis.

Definition 2.1. Let $\mathcal{O}$ be an order ideal of terms in $\mathbb{T}^n$ whose residue classes form a K-vector space basis of P/I.

(1) For $i = 1, \dots, m$, let $\mathcal{O}_i \subseteq \mathcal{O}$ be an order ideal. Then the set $\mathcal{O}_F = \mathcal{O}_1 \cdot f_1 \cup \dots \cup \mathcal{O}_m \cdot f_m$ is called an *F-order ideal*. Its elements, i.e. products of the form tf_i with $t \in \mathcal{O}_i$ will be called *F-terms*.

(2) If $\mathcal{O}_F = \mathcal{O}_1 \cdot f_1 \cup \dots \cup \mathcal{O}_m \cdot f_m$ is an F-order ideal whose residue classes form a K-vector space basis of $J/(I \cap J)$, we say that the ideal I has an *$\mathcal{O}_F$-subideal border basis*.

Notice that an F-term may be viewed as a generalization of the usual notion of term by using $F = \{1\}$. Similarly, F-order ideals generalize the usual order ideals. It is natural to ask whether every ideal I supporting an $\mathcal{O}$-border bases has an $\mathcal{O}_F$-subideal border basis for some F-order ideal $\mathcal{O}_F$. The next proposition answers this positively.

Proposition 2.2. *Let $I \subset P$ be a 0-dimensional ideal, and let $J = \langle f_1, \dots, f_m \rangle \subset P$ be any ideal.*

(1) *Given an order ideal $\mathcal{O} \subset \mathbb{T}^n$ whose residue classes generate the K-vector space P/I, there exists an order ideal $\widetilde{\mathcal{O}} \subseteq \mathcal{O}$ whose residue classes from a K-vector space basis of P/I.*

(2) *Let $U \subset P^m$ be a P-submodule, and let $\mathcal{O}_1, \dots, \mathcal{O}_m$ be order ideals in $\mathbb{T}^n$ such that the residue classes of $\mathcal{O}_1 e_1 \cup \dots \cup \mathcal{O}_m e_m$ generate the K-vector space P^m/U. Then there exist order ideals $\widetilde{\mathcal{O}_i} \subseteq \mathcal{O}_i$ such that the residue classes of $\widetilde{\mathcal{O}}_1 e_1 \cup \dots \cup \widetilde{\mathcal{O}}_m e_m$ form a K-vector space basis of P^m/U.*

(3) *If $\mathcal{O}$ is an order ideal whose residue classes from a K-vector space basis of P/I then there exist order ideals $\mathcal{O}_i \subseteq \mathcal{O}$ such that the residue classes of $\mathcal{O}_F = \mathcal{O}_1 f_1 \cup \dots \cup \mathcal{O}_m f_m$ are a K-vector space basis of $J/(I \cap J)$. In other words, the ideal I has an $\mathcal{O}_F$-subideal border basis.*

Proof. First we show (1). We construct the order ideal $\widetilde{\mathcal{O}}$ inductively. To this end, we choose a degree compatible term ordering σ and order $\mathcal{O} = \{t_1, \dots, t_\mu\}$ such that $t_1 <_\sigma \dots <_\sigma t_\mu$. In particular, we have $t_1 = 1$. Since $I \subset P$, we can start by putting t_1 into $\widetilde{\mathcal{O}}$ and removing it from $\mathcal{O}$.

For the induction step, we consider the σ-smallest term t_i which is still in $\mathcal{O}$. If the residue class of t_i in P/I is K-linearly dependent on the residue classes of the elements in $\widetilde{\mathcal{O}}$, we cancel t_i and all of its multiples in $\mathcal{O}$. Each of these terms can be rewritten modulo I as a linear combination of smaller terms w.r.t. σ. If the residue class of t_i in P/I is K-linearly independent of the residue classes of the elements in $\widetilde{\mathcal{O}}$, we append t_i to $\widetilde{\mathcal{O}}$ and remove it from $\mathcal{O}$. In this way, the residue classes of the elements of $\widetilde{\mathcal{O}}$ are always K-linearly independent in P/I, and every element of $\mathcal{O}$ can be rewritten modulo I as a K-linear combination of the elements of the final set $\widetilde{\mathcal{O}}$.

Now we show (2). Let $\overline{U}$ be the idealization of U in $\overline{P} = K[x_1, \dots, x_n, e_1, \dots, e_m]$ (see [10], Section 4.7.B). The residue classes of the elements of the order ideal $\overline{\mathcal{O}} = \mathcal{O}_1 e_1 \cup \cdots \cup \mathcal{O}_m e_m$ generate the K-vector space $\overline{P}/\overline{U}$. Now it suffices to apply (1) and to note that every order subideal of $\overline{\mathcal{O}}$ has the indicated form.

Finally, we prove (3). Consider the P-linear map $\varphi : P^m \longrightarrow P/I$ defined by $e_i \mapsto f_i + I$. Its image is the ideal $(I+J)/I$. Let $U = \ker(\varphi)$. Then φ induces an isomorphism $\bar{\varphi} : P^m/U \cong (I+J)/I$. To get a system of generators of P^m/U, it suffices to find a system of generators of the ideal $(I+J)/I = \langle f_1 + I, \dots, f_m + I\rangle$. As a vector space, this ideal is generated by $\mathcal{O} \cdot (f_1 + I) \cup \cdots \cup \mathcal{O} \cdot (f_m + I)$. The preimages of these generators are the elements of $\mathcal{O}e_1 \cup \cdots \cup \mathcal{O}e_m$. Now an application of (2) finishes the proof. □

Based on this proposition, we can construct an F-order ideal such that a given ideal I has an $\mathcal{O}_F$-subideal border basis. The following example illustrates the method.

Example 2.3. Let $P = \mathbb{Q}[x, y]$, let $I = \langle x^2 - x,\, y^2 - y\rangle$, let $\mathcal{O} = \{1,\, x,\, y,\, xy\}$, and let $J = \langle x + y\rangle$. Then I has an $\mathcal{O}$-border basis and therefore also an $\mathcal{O}_F$-subideal border basis w.r.t. $F = \{f\}$ for $f = x + y$.

To construct a suitable F-order ideal, we start with $\mathcal{O}_F = \{1 \cdot f\}$. Then we put $x \cdot f$ and $y \cdot f$ into $\mathcal{O}_F$, since we have $x \cdot f \equiv xy + x$ and $y \cdot f \equiv xy + y$ modulo I, and since $\{x + y,\, xy + x,\, xy + y\}$ is $\mathbb{Q}$-linearly independent in P/I. Next $xy \cdot f \equiv 2xy \equiv x \cdot f + y \cdot f - 1 \cdot f$ implies that we are done. The result is that $\mathcal{O}_F = \{f,\, xf,\, yf\}$ is an F-order ideal for which I has an $\mathcal{O}_F$-subideal border basis.

At this point it is time to explain the choice of the term "subideal border basis" in the above definition.

Definition 2.4. Let $F = \{f_1, \dots, f_m\} \subset P \setminus \{0\}$, and let $\mathcal{O}_F = \mathcal{O}_1 f_1 \cup \cdots \cup \mathcal{O}_m f_m$ be an F-order ideal. We write $\mathcal{O}_F = \{t_1 f_{\alpha_1}, \dots, t_\mu f_{\alpha_\mu}\}$ with $\alpha_i \in \{1, \dots, m\}$ and $t_i \in \mathcal{O}_{\alpha_i}$.

(1) The set of polynomials $\partial \mathcal{O}_F = (x_1 \mathcal{O}_F \cup \cdots \cup x_n \mathcal{O}_F) \setminus \mathcal{O}_F$ is called the *border* of $\mathcal{O}_F$.
(2) Let $\partial \mathcal{O}_F = \{b_1 f_{\beta_1}, \dots, b_\nu f_{\beta_\nu}\}$. A set of polynomials $G = \{g_1, \dots, g_\nu\}$ is called an *$\mathcal{O}_F$-subideal border prebasis* if $g_j = b_j f_{\beta_j} - \sum_{i=1}^{\mu} c_{ij} t_i f_{\alpha_i}$ with $c_{1j}, \dots, c_{\mu j} \in K$ for $j = 1, \dots, \nu$.
(3) An $\mathcal{O}_F$-subideal border prebasis G is called an *$\mathcal{O}_F$-subideal border basis* of an ideal I if G is contained in I and the residue classes of the elements of $\mathcal{O}_F$ form a K-vector space basis of $J/(I \cap J)$.

In this terminology, the last part of the preceding proposition can be rephrased as follows.

Corollary 2.5. *Let $\mathcal{O}$ be an order ideal in $\mathbb{T}^n$, and let $I \subset P$ be a 0-dimensional ideal which has an $\mathcal{O}$-border basis. Then I has an $\mathcal{O}_F$-subideal border basis for every ideal $J = \langle f_1, \ldots, f_m \rangle$ and $F = \{f_1, \ldots, f_m\} \subset P \setminus \{0\}$.*

In the setting of Example 2.3, the $\mathcal{O}_F$-subideal border basis of I can be constructed as follows.

Example 2.6. The border of the F-order ideal $\mathcal{O}_F = \{f, xf, yf\}$ is $\partial\mathcal{O}_F = \{x^2f, xyf, y^2f\}$. We compute modulo I and find $x^2f \equiv xf$, $xyf \equiv xf + yf - f$, and $y^2f \equiv yf$. Therefore the set $G = \{x^2f - xf,\, xyf - xf - yf + f,\, y^2f - yf\}$ is an $\mathcal{O}_F$-subideal border basis of I.

If an ideal has an $\mathcal{O}_F$-subideal border basis, the elements of this basis are uniquely determined. This follows exactly as in the case $J = \langle 1 \rangle$, i.e. the case of the usual border bases (see [10], 6.4.17 and 6.4.18). Notice, however, that a set of polynomials may be an F-order ideal in several different ways. This is illustrated by the following example.

Example 2.7. Let $P = \mathbb{Q}[x, y]$, let $I = \langle x^2 - x,\, y^2 - y \rangle$, and let $J = \langle x,\, y \rangle$. Clearly, the ideal I has an $\mathcal{O}$-border basis for $\mathcal{O} = \{1,\, x,\, y,\, xy\}$, namely the set $G = \{x^2 - x,\, x^2y - xy,\, xy^2 - xy,\, y^2 - y\}$. Hence the ideal I also has an $\mathcal{O}_F$-subideal border basis for $F = \{x,\, y\}$. Here we can use both $\mathcal{O}_F = \{1, y\} \cdot x \cup \{1\} \cdot y$ and $\mathcal{O}_F = \{1\} \cdot x \cup \{1, x\} \cdot y$.

This example shows also another phenomenon: an F-term can simultaneously be contained in $\mathcal{O}_F$ and in $\partial\mathcal{O}_F$. For instance, if we use $\mathcal{O}_F = \{1, y\} \cdot x \cup \{1\} \cdot y$, the term xy is both contained in $\{1, y\} \cdot x$ and in the border of $\{1\} \cdot y$. The resulting subideal border basis will contain the polynomial $xy - xy = 0$.

Finally, we give an example where a term is in $\partial\mathcal{O}_F$ in two different ways, so that a subideal border basis polynomial is repeated.

Example 2.8. Let $I = \langle x^2 - x,\, y^2 - y,\, xy \rangle \subseteq \mathbb{Q}[x, y]$, and let $J = \langle x,\, y \rangle \subset \mathbb{Q}[x, y]$. Then the subideal border basis of I with respect to $\mathcal{O}_F = \{1\} \cdot x \cup \{1\} \cdot y$ is $G = \{x^2 - x,\, xy, xy, y^2 - y\}$ where xy appears both in $\partial\{1\} \cdot x$ and in $\partial\{1\} \cdot y$.

3. The Subideal Border Division Algorithm

A central result in the construction of any Gröbner-basis-like theory is a suitable version of the division algorithm (for the classical case, see for instance [9], Thm. 1.6.4 and specifically for border bases, see [10], Prop. 6.4.11). Before we can present a subideal border basis version, we need a few additional definitions.

Definition 3.1. Let $F = \{f_1, \ldots, f_m\} \subset P \setminus \{0\}$, and let $\mathcal{O}_F$ be an F-order ideal.

1. The *first border closure* of $\mathcal{O}_F$ is $\overline{\partial\mathcal{O}_F} = \mathcal{O}_F \cup \partial\mathcal{O}_F$.
2. For every $k \geq 1$, we inductively define the $(k+1)^{st}$ *border* of $\mathcal{O}_F$ by $\partial^{k+1}\mathcal{O}_F = \partial(\overline{\partial^k\mathcal{O}_F})$ and the $(k+1)^{st}$ *border closure* of $\mathcal{O}_F$ by $\overline{\partial^{k+1}\mathcal{O}_F} = \overline{\partial^k\mathcal{O}_F} \cup \partial^{k+1}\mathcal{O}_F$.
3. Finally, we let $\partial^0\mathcal{O}_F = \overline{\partial^0\mathcal{O}_F} = \mathcal{O}_F$.

Using these higher borders, the set $\mathbb{T}^n f_1 \cup \cdots \cup \mathbb{T}^n f_m$ is partitioned as follows.

Proposition 3.2. *Let $F = \{f_1, \dots, f_m\} \subset P \setminus \{0\}$, and let $\mathcal{O}_F$ be an F-order ideal.*

1. *For every $k \geq 0$, we have a disjoint union $\overline{\partial^k \mathcal{O}_F} = \bigcup_{i=0}^{k} \partial^i \mathcal{O}_F$.*
2. *For every $k \geq 0$, we have $\overline{\partial^k \mathcal{O}_F} = \mathbb{T}^n_{\leq k} \cdot \mathcal{O}_F$.*
3. *For every $k \geq 1$, we have $\partial^k \mathcal{O}_F = \mathbb{T}^n_k \cdot \mathcal{O}_F \setminus \mathbb{T}^n_{<k} \cdot \mathcal{O}_F$.*
4. *We have $\bigcup_{i=0}^{m} \mathbb{T}^n \cdot f_i = \bigcup_{j=0}^{\infty} \partial^j \mathcal{O}_F$, where the right-hand side is a disjoint union.*
5. *Any F-term $tf_i \in \mathbb{T}^n \cdot f_i \setminus \mathcal{O}_F$ is divisible by an F-term in $\partial \mathcal{O}_F$.*

Proof. First we show (1) by induction on k. For $k = 0$, the claim follows from the definition. For $k = 1$, we have $\overline{\partial^1 \mathcal{O}_F} = \partial^0 \mathcal{O}_F \cup \partial^1 \mathcal{O}_F$ by Definition 3.1.a. Inductively, it follows that $\overline{\partial^{k+1} \mathcal{O}_F} = \overline{\partial^k \mathcal{O}_F} \cup \partial^{k+1} \mathcal{O}_F = \bigcup_{i=0}^{k+1} \partial^i \mathcal{O}_F$. This is a disjoint union, since $\partial^{k+1} \mathcal{O}_F \cap \overline{\partial^k \mathcal{O}_F} = \emptyset$ in each step.

Next we prove claim (2). Again we proceed by induction on k, the case $k = 0$ being obviously true. Inductively, we have $\overline{\partial^{k+1} \mathcal{O}_F} = \overline{\partial^k \mathcal{O}_F} \cup \partial^{k+1} \mathcal{O}_F = \mathbb{T}^n_{\leq k} \cdot \mathcal{O}_F \cup \mathbb{T}^n_1 \cdot (\mathbb{T}^n_{\leq k} \cdot \mathcal{O}_F) = \mathbb{T}^n_{\leq k+1} \cdot \mathcal{O}_F$.

Claim (3) is a consequence of (2) and the equality $\partial^k \mathcal{O}_F = \overline{\partial^k \mathcal{O}_F} \setminus \overline{\partial^{k-1} \mathcal{O}_F}$. The fourth claim follows from the observation that, by (2), every F-term is in $\overline{\partial^k \mathcal{O}_F}$ for some $k \geq 0$.

Finally, claim (5) holds because (4) implies that $tf_i \in \partial^k \mathcal{O}_F$ for some $k \geq 1$, and by (3) this is equivalent to the existence of a factorization $t = t't''$ where $\deg(t') = k - 1$ and $t'' f_i \in \partial \mathcal{O}_F$. □

In view of this result, the following definition appears natural.

Definition 3.3. Let $F = \{f_1, \dots, f_m\} \subset P \setminus \{0\}$, and let $\mathcal{O}_F$ be an F-order ideal.

1. For an F-term $tf_i \in \mathcal{O}_F$, we define $\operatorname{ind}_{\mathcal{O}_F}(tf_i) = \min\{k \geq 0 \mid tf_i \in \overline{\partial^k \mathcal{O}_F}\}$ and call it the *$\mathcal{O}_F$-index* of tf_i.
2. Given a non-zero polynomial $f \in J$, we write $f = p_1 f_1 + \cdots + p_m f_m$ with $p_i \in P$ and we let $\mathcal{P} = (p_1 f_1, \dots, p_m f_m)$. Then the number
$$\operatorname{ind}_{\mathcal{O}_F}(\mathcal{P}) = \max\{\operatorname{ind}_{\mathcal{O}_F}(tf_i) \mid i \in \{1, \dots, m\},\ t \in \operatorname{Supp}(p_i)\}$$
is called the *$\mathcal{O}_F$-index* of the *representation* $\mathcal{P}$ of f.

In other words, the $\mathcal{O}_F$-index of tf_i is the unique number $k \geq 0$ such that $tf_i \in \partial^k \mathcal{O}_F$. Note that the $\mathcal{O}_F$-index of a polynomial $f \in J$ depends on the representation of f in terms of the generators of J. It is not clear how to find a representation $\mathcal{P}$ which yields the smallest $\operatorname{ind}_{\mathcal{O}_F}(\mathcal{P})$. Using the Subideal Border Division Algorithm, we shall address this point below.

The following proposition collects some basic properties of the $\mathcal{O}_F$-index.

Proposition 3.4. *Let $F = \{f_1, \dots, f_m\} \subset P \setminus \{0\}$, and let $\mathcal{O}_F$ be an F-order ideal.*

1. *For an F-term $tf_i \in \mathbb{T}^n \cdot f_i$, the number $k = \operatorname{ind}_{\mathcal{O}_F}(tf_i)$ is the smallest natural number such that there exists a factorization $t = t't''$ with a term $t' \in \mathbb{T}^n$ of degree k and with $t'' f_i \in \mathcal{O}_F$.*
2. *Given $t \in \mathbb{T}^n$ and an F-term $t' f_i \in \mathbb{T}^n \cdot f_i$, we have*
$$\operatorname{ind}_{\mathcal{O}_F}(t\, t' f_i) \leq \deg(t) + \operatorname{ind}_{\mathcal{O}_F}(t' f_i).$$

(3) *For $f, g \in J\setminus\{0\}$ such that $f+g \neq 0$, we write $f = p_1 f_1 + \cdots + p_m f_m$ and $g = q_1 f_1 + \cdots + q_m f_m$ with $p_i, q_j \in P$, and we let $\mathcal{P} = (p_1 f_1, \dots, p_m f_m)$ and $\mathcal{Q} = (q_1 f_1, \dots, q_m f_m)$. Then we have*

$$\mathrm{ind}_{\mathcal{O}_F}(\mathcal{P} + \mathcal{Q}) \leq \max\{\mathrm{ind}_{\mathcal{O}_F}(\mathcal{P}), \mathrm{ind}_{\mathcal{O}_F}(\mathcal{Q})\}.$$

(4) *Given $f \in J \setminus \{0\}$, we write $f = p_1 f_1 + \cdots + p_m f_m$ with $p_i \in P$ and let $\mathcal{P} = (p_1 f_1, \dots, p_m f_m)$. For every $g \in P \setminus \{0\}$, we then have*

$$\mathrm{ind}_{\mathcal{O}_F}(g\mathcal{P}) \leq \deg(g) + \mathrm{ind}_{\mathcal{O}_F}(\mathcal{P}).$$

Proof. The first claim follows from Prop. 3.2. The second claim follows from the first. The third claim is a consequence of the fact that every F-term appearing in $\mathcal{P} + \mathcal{Q}$ appears in $\mathcal{P}$ or $\mathcal{Q}$. The last claim follows from (2) and the observation that $g\mathcal{P}$ is a K-linear combination of tuples $t\mathcal{P}$ with $t \in \mathrm{Supp}(g)$. □

Now we have collected enough material to formulate and prove the subideal version of the Border Division Algorithm.

Algorithm 3.5. (The Subideal Border Division Algorithm)
Let $F = \{f_1, \dots, f_m\} \subset P \setminus \{0\}$, let $\mathcal{O}_F = \{t_1 f_{\alpha_1}, \dots, t_\mu f_{\alpha_\mu}\}$ be an F-order ideal where $\alpha_i \in \{1, \dots, m\}$ and $t_i \in \mathcal{O}_{\alpha_i}$, let $\partial\mathcal{O}_F = \{b_1 f_{\beta_1}, \dots, b_\nu f_{\beta_\nu}\}$ be its border, and let $\{g_1, \dots, g_\nu\}$ be an $\mathcal{O}_F$-subideal border prebasis, where $g_j = b_j f_{\beta_j} - \sum_{i=1}^{\mu} c_{ij} t_i f_{\alpha_i}$ with $c_{1j}, \dots, c_{\mu j} \in K$ for $j = 1, \dots, \nu$. Given a polynomial $f \in J$, we write $f = p_1 f_1 + \cdots + p_m f_m$ and consider the following instructions.

D1 *Let $h_1 = \cdots = h_\nu = 0$, $c_1 = \cdots = c_\mu = 0$, and $\mathcal{Q} = (q_1 f_1, \dots, q_m f_m)$ with $q_i = p_i$ for $i = 1, \dots, m$.*

D2 *If $\mathcal{Q} = (0, \dots, 0)$ then return $(h_1, \dots, h_\nu, c_1, \dots, c_\mu)$ and stop.*

D3 *If $\mathrm{ind}_{\mathcal{O}_F}(\mathcal{Q}) = 0$ then find $c_1, \dots, c_\mu \in K$ such that $q_1 f_1 + \cdots + q_m f_m = c_1 t_1 f_{\alpha_1} + \cdots + c_\mu t_\mu f_{\alpha_\mu}$. Return $(h_1, \dots, h_\nu, c_1, \dots, c_\mu)$ and stop.*

D4 *If $\mathrm{ind}_{\mathcal{O}_F}(\mathcal{Q}) > 0$ then determine the smallest index $i \in \{1, \dots, m\}$ such that there exists a term $t \in \mathrm{Supp}(q_i)$ with $\mathrm{ind}_{\mathcal{O}_F}(t f_i) = \mathrm{ind}_{\mathcal{O}_F}(\mathcal{Q})$. Choose such a term t. Let $a \in K$ be the coefficient of t in q_i. Next, determine the smallest index $j \in \{1, \dots, \nu\}$ such that t factors as $t = t' t''$ with a term t' of degree $\mathrm{ind}_{\mathcal{O}_F}(t f_i) - 1$ and with $t'' f_i = b_j f_{\beta_j} \in \partial\mathcal{O}_F$. Subtract the tuple corresponding to the representation*

$$a\, t'\, g_j = a\, t'\, b_j f_{\beta_j} - \sum_{i=1}^{\mu} c_{ij}\, a\, t'\, t_i f_{\alpha_i}$$

from $\mathcal{Q}$, add at' to h_j, and continue with step **D2**.

This is an algorithm which returns a tuple $(h_1, \dots, h_\nu, c_1, \dots, c_\mu) \in P^\nu \times K^\mu$ such that

$$f = h_1 g_1 + \cdots + h_\nu g_\nu + c_1 t_1 f_{\alpha_1} + \cdots + c_\mu t_\mu f_{\alpha_\mu}$$

and $\deg(h_i) \leq \mathrm{ind}_{\mathcal{O}_F}(\mathcal{P}) - 1$ for $\mathcal{P} = (p_1 f_1, \dots, p_m f_m)$ and for all $i \in \{1, \dots, \nu\}$ with $h_i \neq 0$. This representation does not depend on the choice of the term t in step **D4**.

Proof. First we show that all steps can be executed. In step **D3**, the condition $\mathrm{ind}_{\mathcal{O}_F}(\mathcal{Q}) = 0$ implies that all F-terms $t f_i$ with $t \in \mathrm{Supp}(q_i)$ are contained in $\mathcal{O}_F$. In step **D4**, the definition of $\mathrm{ind}_{\mathcal{O}_F}(\mathcal{Q})$ implies that a term t of the desired kind exists. By Proposition 3.4.1, this term t has a factorization $t = t' t''$ with the desired properties.

Next we prove termination by showing that step **D4** is performed only finitely many times. Let us investigate the subtraction of the representation of $at'g_j$ from $\mathcal{Q}$. By the choice of t', the $\mathcal{O}_F$-index of $t'b_jf_{\beta_j}$ is $\deg(t')$ more than the $\mathcal{O}_F$-index of $b_jf_{\beta_j}$. By Prop. 3.4.b, this is the maximal increase, and the $\mathcal{O}_F$-index of the other F-terms in the representation of $at'g_j$ is smaller than $\operatorname{ind}_{\mathcal{O}_F}(\mathcal{Q})$. Thus the number of F-terms in $\mathcal{Q}$ of maximal $\mathcal{O}_F$-index decreases by the subtraction, and after finitely many steps the algorithm reaches step **D2** or **D3** and stops.

Finally, we prove correctness. To do so, we show that the equality

$$f = q_1f_1 + \cdots + q_mf_m + h_1g_1 + \cdots + h_\nu g_\nu + c_1t_1f_{\alpha_1} + \cdots + c_\mu t_\mu f_{\alpha_\mu}$$

is an invariant of the algorithm. It is satisfied at the end of step **D1**. The constants $c_1, \dots, c_\mu$ are only changed in step **D3**. In this case the contribution $q_1f_1 + \cdots + q_mf_m$ to the above equality is replaced by the equal contribution $c_1t_1f_{\alpha_1} + \cdots + c_\mu t_\mu f_{\alpha_\mu}$. The tuple $\mathcal{Q}$ is only changed in step **D4**. There the subtraction of the representation of $at'g_j$ from $\mathcal{Q}$ and the corresponding change in $q_1f_1 + \cdots + q_mf_m$ are compensated by the addition of at' to h_j and the corresponding change in $h_1g_1 + \cdots + h_\nu g_\nu$. When the algorithm stops, we have $q_1 = \cdots = q_m = 0$. This proves the claimed representation of f. Moreover, only terms of degree $\deg(t') \le \operatorname{ind}_{\mathcal{O}_F}(\mathcal{Q}) - 1 \le \operatorname{ind}_{\mathcal{O}_F}(\mathcal{P}) - 1$ are added to h_j.

The additional claim that the result of the algorithm does not depend on the choice of t in step **D4** follows from the observation that tf_i is replaced by F-terms of strictly smaller $\mathcal{O}_F$-index. Thus the different executions of step **D4** corresponding to the reduction of several F-terms of maximal $\mathcal{O}_F$-index in $\mathcal{Q}$ do not interfere with one another, and the final result – after all those F-terms have been rewritten – is independent of the order in which they are taken care of. □

Notice that in step **D4** the algorithm uses a term t which is not uniquely determined. Also there may be several factorizations of t. We choose the indices i and j minimally to determine this step of the algorithm uniquely, but this particular choice is not forced upon us. Moreover, it is clear that the result of the division depends on the numbering of the elements of $\partial\mathcal{O}_F$.

As indicated above, the Subideal Border Division Algorithm has important implications. The following corollaries comprise a few of them.

Corollary 3.6. (Subideal Border Bases and Special Generation)
In the setting of the algorithm, let $I = \langle G \rangle$. Then the set G is an $\mathcal{O}_F$-subideal border basis of I if and only if one of the following equivalent conditions is satisfied.

- (A_1) *For every non-zero polynomial $f \in I \cap J$ with a representation $f = p_1f_1 + \cdots + p_mf_m$ and $\mathcal{P} = (p_1f_1, \dots, p_mf_m)$, there exist polynomials $h_1, \dots, h_\nu \in P$ such that $f = h_1g_1 + \cdots + h_\nu g_\nu$ and $\deg(h_i) \le \operatorname{ind}_{\mathcal{O}_F}(\mathcal{P}) - 1$ whenever $h_ig_i \ne 0$.*
- (A_2) *For every non-zero polynomial $f \in I \cap J$ with a representation $f = p_1f_1 + \cdots + p_mf_m$ and $\mathcal{P} = (p_1f_1, \dots, p_mf_m)$, there exist $h_1, \dots, h_\nu \in P$ such that $f = h_1g_1 + \cdots + h_\nu g_\nu$ and $\max\{\deg(h_i) \mid i \in \{1, \dots, \nu\},\ h_ig_i \ne 0\} = \operatorname{ind}_{\mathcal{O}_F}(\mathcal{P}) - 1$.*

Proof. First we show that (A_1) holds if G is an $\mathcal{O}_F$-border basis. The Subideal Border Division Algorithm computes a representation $f = h_1g_1 + \cdots + h_\nu g_\nu + c_1t_1f_{\alpha_1} + \cdots + c_\mu t_\mu f_{\alpha_\mu}$ with $h_1, \dots, h_\nu \in P$ and $c_1, \dots, c_\mu \in K$ such that $\deg(h_i) \le$

$\operatorname{ind}_{\mathcal{O}_F}(\mathcal{P})-1$ for $i=1,\dots,\nu$. Then $c_1t_1f_{\alpha_1}+\cdots+c_\mu t_\mu f_{\alpha_\mu}\equiv 0$ modulo I, and the hypothesis implies $c_1=\cdots=c_\mu=0$.

Next we prove that (A_1) implies (A_2). If $\deg(h_i)<\operatorname{ind}_{\mathcal{O}_F}(\mathcal{P})-1$, then Prop. 3.4.2 shows that the $\mathcal{O}_F$-index of every representation of h_ig_i is at most $\deg(h_i)+1$ and hence smaller than $\operatorname{ind}_{\mathcal{O}_F}(\mathcal{P})$. By Prop. 3.4.4, there has to be at least one number $i\in\{1,\dots,\nu\}$ such that $\deg(h_i)=\operatorname{ind}_{\mathcal{O}_F}(\mathcal{P})-1$.

Finally, we assume (A_2) and show the subideal border basis property. Let $c_1,\dots,c_\mu\in K$ satisfy $c_1t_1f_{\alpha_1}+\cdots+c_\mu t_\mu f_{\alpha_\mu}\in I\cap J$. Then either $f=c_1t_1f_{\alpha_1}+\cdots+c_\mu t_\mu f_{\alpha_\mu}$ equals the zero polynomial or not. In the latter case we apply (A_2) and obtain a representation $f=h_1g_1+\cdots+h_\nu g_\nu$ with $h_1,\dots,h_\nu\in P$. Since $f\neq 0$, we have $\max\{\deg(h_i)\mid i\in\{1,\dots,\nu\},\, h_ig_i\neq 0\}\geq 0$. But $\operatorname{ind}_{\mathcal{O}_F}(\mathcal{P})-1=-1$ is in contradiction to the second part of (A_2). Hence we must have $f=0$. Thus $I\cap J\cap\langle\mathcal{O}_F\rangle_K=0$, i.e. the set G is an $\mathcal{O}_F$-subideal border basis of I. □

Definition 3.7. In the setting of the algorithm, let $\mathcal{G}=(g_1,\dots,g_\nu)$. Then the polynomial

$$\operatorname{NR}_{\mathcal{O}_F,\mathcal{G}}(\mathcal{P})=c_1t_1f_{\alpha_1}+\cdots+c_\mu t_\mu f_{\alpha_\mu}$$

is called the *normal remainder* of the representation $\mathcal{P}=(p_1f_1,\dots,p_mf_m)$ of f with respect to $\mathcal{G}$.

Clearly, the normal remainder depends on the choice of the representation $\mathcal{P}$. It has the following application.

Corollary 3.8. *In the setting of the algorithm, the residue classes of the elements of $\mathcal{O}_F$ generate the image of the ideal J in $P/\langle G\rangle$ as a K-vector space.*

In other words, the residue class of every polynomial $f\in J$ can be represented as a K-linear combination of the residue classes $\{\bar t_1\bar f_{\alpha_1},\dots,\bar t_\mu\bar f_{\alpha_\mu}\}$. Indeed, such a representation can be found by computing the normal remainder $\operatorname{NR}_{\mathcal{O}_F,\mathcal{G}}(\mathcal{P})$ for $\mathcal{G}=(g_1,\dots,g_\nu)$ and the representation $\mathcal{P}=(p_1f_1,\dots,p_mf_m)$ of $f=p_1f_1+\cdots+p_mf_m$.

Proof. By the algorithm, every $f\in J$ can be represented in the form $f=h_1g_1+\cdots+h_\nu g_\nu+c_1t_1f_{\alpha_1}+\cdots+c_\mu t_\mu f_{\alpha_\mu}$, where $h_1,\dots,h_\nu\in P$ and $c_1,\dots,c_\mu\in K$. Forming residue classes modulo $\langle G\rangle$ yields the claim. □

Our last corollary provides another motivation for the name "subideal border basis".

Corollary 3.9. *In the setting of the algorithm, let G be an $\mathcal{O}_F$-subideal border basis of an ideal $I\subset P$. Then G generates the ideal $I\cap J$.*

Proof. By definition, we have $\langle g_1,\dots,g_\nu\rangle\subseteq I\cap J$. To prove the converse inclusion, let $f\in I\cap J$. Using the Subideal Border Division Algorithm, the polynomial f can be expanded as $f=h_1g_1+\cdots+h_\nu g_\nu+c_1t_1f_{\alpha_1}+\cdots+c_\mu t_\mu f_{\alpha_\mu}$, where $h_1,\dots,h_\nu\in P$ and $c_1,\dots,c_\mu\in K$. This implies the equality of residue classes $0=\bar f=c_1\bar t_1\bar f_{\alpha_1}+\cdots+c_\mu\bar t_\mu\bar f_{\alpha_\mu}$ in P/I. By assumption, the residue classes $\bar t_1\bar f_{\alpha_1},\dots,\bar t_\mu\bar f_{\alpha_\mu}$ form a K-vector space basis of $(I+J)/I$. Hence $c_1=\cdots=c_\mu=0$, and the expansion of f yields $f=h_1g_1+\cdots+h_\nu g_\nu\in\langle G\rangle$. □

4. The Subideal Version of the BM-Algorithm

Let K be a field, let $P = K[x_1, \dots, x_n]$ be the polynomial ring in n indeterminates over K, equipped with the standard grading, and let $\mathbb{T}^n$ be the monoid of terms in P. Given a finite set of points $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq K^n$, we let $\mathrm{eval} : P \longrightarrow K^s$ be the evaluation map $\mathrm{eval}(f) = (f(p_1), \dots, f(p_s))$ associated to $\mathbb{X}$. It is easy to adjust the Buchberger-Möller Algorithm (BM-Algorithm) so that it computes a border basis of the *vanishing ideal*

$$I_{\mathbb{X}} = \langle f \in P \mid f(p_1) = \cdots = f(p_s) = 0 \rangle = \ker(\mathrm{eval}) \subseteq P$$

of $\mathbb{X}$. Since we use a version which differs slightly from the standard formulation (see for instance [4] or [10], Thm. 6.3.10), let us briefly recall its main steps.

Algorithm 4.1. (BM-Algorithm for Border Bases)
Let $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq K^n$ be a set of points given by their coordinates, and let σ be a degree compatible term ordering on $\mathbb{T}^n$. The following instructions define an algorithm which computes the order ideal $\mathcal{O}_\sigma(I) = \mathbb{T}^n \setminus \mathrm{LT}_\sigma(I_{\mathbb{X}})$ and the $\mathcal{O}_\sigma(I_{\mathbb{X}})$-border basis G of $I_{\mathbb{X}}$.

B1 *Let $d = 0$, $\mathcal{O} = \{1\}$, $G = \emptyset$, and $\mathcal{M} = (1, \dots, 1)^{\mathrm{tr}} \in \mathrm{Mat}_{s,1}(K)$.*

B2 *Increase d by one and let $L = [t_1, \dots, t_\ell]$ be the list of all terms of degree d in $\partial\mathcal{O}$, ordered decreasingly w.r.t. σ. If $L = \emptyset$, return $(\mathcal{O}, G)$ and stop.*

B3 *Form the matrix $\mathcal{A} = (\mathrm{eval}(t_1) \mid \cdots \mid \mathrm{eval}(t_\ell) \mid \mathcal{M})$ and compute a matrix $\mathcal{B}$ whose rows are a basis of the kernel of $\mathcal{A}$.*

B4 *Reduce $\mathcal{B}$ to a matrix $\mathcal{C} = (c_{ij}) \in \mathrm{Mat}_{k,\ell+m}(K)$ in row echelon form.*

B5 *For all $j \in \{1, \dots, \ell\}$ such that there exists an $i \in \{1, \dots, k\}$ with pivot index $\nu(i) = j$, append the polynomial*

$$t_j + \sum_{j'=j+1}^{\ell} c_{ij'} t_{j'} + \sum_{j'=\ell+1}^{\ell+m} c_{ij'} u_{j'}$$

to the list G, where $u_{j'}$ is the $(j' - \ell)^{\mathrm{th}}$ element of $\mathcal{O}$.

B6 *For all $j = \ell, \ell-1, \dots, 1$ such that the j^{th} column of $\mathcal{C}$ contains no pivot element, append the term t_j as a new first element to $\mathcal{O}$, append the column $\mathrm{eval}(t_j)$ as a new first column to $\mathcal{M}$, and continue with step* **B2**.

The proof of this modified version is simply obtained by combining all the iterations of the usual BM-Algorithm corresponding to terms of degree d into one "block". The fact that we put the terms of degree d in $\partial\mathcal{O}$ into L in step **B2** effects the computation of the entire border basis, rather than just the reduced σ-Gröbner basis of $I_{\mathbb{X}}$ (see [5], Thm. 3.3). A further elaboration is beyond the scope of the present paper and is left to the interested reader.

Given $\mathbb{X}$ and a polynomial ideal $J = \langle F \rangle$ with $F = \{f_1, \dots, f_m\} \subset P \setminus \{0\}$, we know that the vanishing ideal $I_{\mathbb{X}}$ has an $\mathcal{O}_\sigma(I)_F$-subideal border basis. The following generalization of the BM-algorithm computes this subideal border basis.

Algorithm 4.2. (Subideal Version of the BM-Algorithm)
Let $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq K^n$ be a set of points given by their coordinates, let σ be a degree compatible term ordering, and let $F = \{f_1, \dots, f_m\} \subset P \setminus \{0\}$ be a set of polynomials which generate an ideal $J = \langle F \rangle$. The following instructions define an algorithm which computes an F-order ideal $\mathcal{O}_\sigma(I)_F$ and the $\mathcal{O}_\sigma(I)_F$-subideal border basis G of $I_{\mathbb{X}}$.

S1 *Let* $d = \min\{\deg(f_1), \dots, \deg(f_m)\} - 1$, $\mathcal{O}_F = \emptyset$, $G = \emptyset$, *and* $\mathcal{M} \in \mathrm{Mat}_{s,0}(K)$.

S2 *Increase* d *by one. Let* $L = [t_1 f_{\alpha_1}, \dots, t_\ell f_{\alpha_\ell}]$ *be the list of all* F*-terms of degree* d *in* $F \cup \partial \mathcal{O}_F$, *with their leading terms ordered decreasingly w.r.t.* σ. *If then* $L = \emptyset$ *and* $d \geq \max\{\deg(f_1), \dots, \deg(f_m)\}$, *return* $(\mathcal{O}_F, G)$ *and stop.*

S3 *Form the matrix* $\mathcal{A} = (\mathrm{eval}(t_1 f_{\alpha_1}) \mid \cdots \mid \mathrm{eval}(t_\ell f_{\alpha_\ell}) \mid \mathcal{M})$ *and compute a matrix* $\mathcal{B}$ *whose rows are a basis of the kernel of* $\mathcal{A}$.

S4 *Reduce* $\mathcal{B}$ *to a matrix* $\mathcal{C} = (c_{ij}) \in \mathrm{Mat}_{k,\ell+m}(K)$ *in reduced row echelon form.*

S5 *For all* $j \in \{1, \dots, \ell\}$ *such that there exists an* $i \in \{1, \dots, k\}$ *with pivot index* $\nu(i) = j$, *append the polynomial*

$$t_j f_{\alpha_j} + \sum_{j'=j+1}^{\ell} c_{ij'} t_{j'} f_{\alpha_{j'}} + \sum_{j'=\ell+1}^{\ell+m} c_{ij'} u_{j'}$$

to the list G, *where* $u_{j'}$ *is the* $(j' - \ell)^{\mathrm{th}}$ *element of* $\mathcal{O}_F$.

S6 *For all* $j = \ell, \ell - 1, \dots, 1$ *such that the* j^{th} *column of* $\mathcal{C}$ *contains no pivot element, append the* F*-term* $t_j f_{\alpha_j}$ *as a new first element to* $\mathcal{O}_F$, *append the column* $\mathrm{eval}(t_j f_{\alpha_j})$ *as a new first column to* $\mathcal{M}$, *and continue with step* **S2**.

Proof. First we show finiteness. When a new degree is started in step **S2**, the matrix $\mathcal{M}$ has $m = \#\mathcal{O}_F$ columns where $\mathcal{O}_F$ is the *current* list of F-terms. In step **S6** we enlarge $\mathcal{M}$ by new first columns which are linearly independent of the other columns. This can happen only finitely many times. Eventually we arrive at a situation where all new columns $\mathrm{eval}(t_i f_{\alpha_i})$ of $\mathcal{A}$ in step **S3** are linearly dependent on the previous columns, and therefore the corresponding column of $\mathcal{C}$ contains a pivot element. Consequently, no elements are appended to $\mathcal{O}_F$ in that degree and we get $L = \emptyset$ in the next degree. Hence the algorithm stops.

Now we show correctness. The columns of $\mathcal{A}$ are the evaluation vectors of F-terms whose leading terms are ordered decreasingly w.r.t. σ. A row $(c_{i1}, \dots, c_{i\,\ell+m})$ of $\mathcal{C}$ corresponds to a linear combination of these F-terms whose evaluation vector is zero. Let $g_1, \dots, g_k$ be the polynomials given by these linear combinations of F-terms. Clearly, we have $g_i \in I_{\mathbb{X}} \cap J$.

The evaluation vectors of the F-terms which are put into $\mathcal{O}_F$ in step **S6** are linearly independent of the evaluation vectors of the F-terms in the previous set $\mathcal{O}_F$ since there is no linear relation leading to a pivot element in the corresponding column of $\mathcal{C}$. Inductively it follows that the evaluation vectors of the F-terms in $\mathcal{O}_F$ are always linearly independent. Henceforth the pivot elements of $\mathcal{C}$ are always in the "new" columns and the polynomials g_i have degree d. By the way the algorithm proceeds, every F-term in the border of the final set $\mathcal{O}_F$ appears in exactly one on the elements of G. All the other summands of a polynomial g_i are in $\mathcal{O}_F$. Hence the final set G is an $\mathcal{O}_F$-subideal border prebasis.

Furthermore, every F-term is either in $\mathcal{O}_F$ or it is a multiple of an F-term in $\partial \mathcal{O}_F$ (cf. Prop. 3.4.5). In the latter case, its evaluation vector can be written as a linear combination of the evaluation vectors of the elements of $\mathcal{O}_F$. Thus the evaluation vectors of the elements of $\mathcal{O}_F$ generate the space of all evaluation vectors of F-terms. Since they are linearly independent, they form a K-basis of that space. Now we use the facts that evaluation yields an isomorphism of K-vector spaces $\overline{\mathrm{eval}} : P/I \longrightarrow K^s$ and that the residue classes of the F-terms generate the K-vector subspace $(I+J)/I$ of P/I to conclude that the residue classes of the F-terms in the final set $\mathcal{O}_F$ form a K-basis of $(I+J)/I$. □

Let us illustrate this algorithm by an example.

Example 4.3. In the polynomial ring $P = \mathbb{Q}[x,y,z]$, we consider the ideal $J = \langle F \rangle$ with $F = \{f_1,\, f_2\}$ given by $f_1 = x^2 - 1$ and $f_2 = y - z$. Let $\sigma = \mathtt{DegRevLex}$.

We want to compute an $\mathcal{O}_F$-subideal border basis of the vanishing ideal of the point set $\mathbb{X} = \{(1,1,1),\,(0,1,1),\,(1,1,0),\,(1,0,1)\}$. Notice that the first point of $\mathbb{X}$ lies on $\mathcal{Z}(f_1, f_2)$, so that we should expect an F-order ideal consisting of three F-terms. Let us follow the steps of the algorithm. (We only list those steps in which something happens.)

S2 Let $d = 1$ and $L = [y-z]$.
S3 Form $\mathcal{A} = (0,0,1,-1)^{\mathrm{tr}}$ and compute $\mathcal{B} = (0)$. (Thus $\mathcal{C} = \mathcal{B}$.)
S6 Let $\mathcal{O}_F = \{y-z\}$ and $\mathcal{M} = (0,0,1,-1)^{\mathrm{tr}}$.
S2 Let $d = 2$ and $L = [x^2-1,\, x(y-z),\, y(y-z),\, z(y-z)]$.

S3 Compute $\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & -1 \end{pmatrix}$ and $\mathcal{B} = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}$.
(Thus $\mathcal{C} = \mathcal{B}$.)

S5 The pivot indices $\nu(1) = 2$ and $\nu(2) = 3$ yield the set $G = \{g_1, g_2\}$ with $g_1 = x(y-z) - (y-z)$ and $g_2 = y(y-z) + z(y-z) - (y-z)$.

S6 We obtain $\mathcal{O}_F = \{x^2-1,\, z(y-z),\, y-z\}$ and $\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$.

S2 Let $d = 3$. We have $L = [x(x^2-1),\, y(x^2-1),\, z(x^2-1),\, xz(y-z),\, yz(y-z),\, z^2(y-z)]$.

S3 Find $\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$

and $\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$. (Thus $\mathcal{C} = \mathcal{B}$.)

S5 Here we obtain $G = \{g_1, \ldots, g_8\}$ where $g_3 = x(x^2-1)$, $g_4 = y(x^2-1)$, $g_5 = z(x^2-1)$, $g_6 = xz(y-z) - z(y-z)$, $g_7 = yz(y-z)$, and finally $g_8 = z^2(y-z) - z(y-z)$.
S6 There are no new non-pivot indices. Hence $\mathcal{O}$ and $\mathcal{M}$ are not changed.
S2 We get $L = \emptyset$ and the algorithm stops.

The result is the F-order ideal $\mathcal{O}_F = \{x^2-1,\, z(y-z),\, y-z\}$ and the $\mathcal{O}_F$-subideal border basis $G = \{g_1, \ldots, g_8\}$ of $I_{\mathbb{X}}$.

5. The Subideal Version of the AVI-Algorithm

From here on we work in the polynomial ring $P = \mathbb{R}[x_1, \ldots, x_n]$ over the field of real numbers. We let $\mathbb{X} = \{p_1, \ldots, p_s\} \subset [-1,1]^n \subset \mathbb{R}^n$ be a finite set of points and $\varepsilon > \tau > 0$ two threshold numbers. (The number ε can be thought of as a measure for error tolerance of the input data points $\mathbb{X}$ and τ is used as a “minimum size” for acceptable leading coefficients of unitary polynomials.)

Let us point out the following notational convention we are using: the “usual” norm of a polynomial $f \in P$ is the Euclidean norm of its coefficient vector and is denoted by $\|f\|$. By “unitary” we mean $\|f\| = 1$. In contrast, by $\|f\|_1$ we mean the sum of the absolute values of the coefficients of f, and the term “$\| \; \|_1$-unitary” is to be interpreted accordingly.

Furthermore, by eval : $P \longrightarrow \mathbb{R}^s$ we denote the evaluation map $\mathrm{eval}(f) = (f(p_1), \ldots, f(p_s))$ associated to $\mathbb{X}$. For the convenience of the reader, we briefly recall the basic structure of the Approximate Vanishing Ideal Algorithm (AVI-algorithm) from [5]. Notice that we skip several technical details and explicit error estimates. The goal of the AVI-algorithm is to compute an approximate border basis, a notion that is defined as follows.

Definition 5.1. Let $\mathcal{O} = \{t_1, \ldots, t_\mu\} \subseteq \mathbb{T}^n$ be an order ideal of terms, let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ be its border, and let $G = \{g_1, \ldots, g_\nu\}$ be an $\mathcal{O}$-border prebasis of the ideal $I = \langle g_1, \ldots, g_\nu \rangle$ in P. Recall that this means that g_j is of the form $g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$ with $c_{ij} \in \mathbb{R}$.

For every pair (i,j) such that b_i, b_j are neighbors in $\partial\mathcal{O}$, we compute the normal remainder $S'_{ij} = \mathrm{NR}_{\mathcal{O},G}(S_{ij})$ of the S-polynomial of g_i and g_j with respect to G. We say that G is an ε*-approximate border basis* of the ideal $I = \langle G \rangle$ if we have $\|S_{ij}\| < \varepsilon$ for all such pairs (i,j).

Moreover, the AVI-algorithm uses the concepts of approximate vanishing, approximate kernel and stabilized reduced row echelon form, for which we refer to [5], Sect. 2 and 3.

Algorithm 5.2. (AVI-Algorithm)
Let $\mathbb{X} = \{p_1, \ldots, p_s\} \subset [-1,1]^n \subset \mathbb{R}^n$ be a set of points as above, and let σ be a degree compatible term ordering. Consider the following sequence of instructions.

- **A1** *Start with lists $G = \emptyset$, $\mathcal{O} = [1]$, a matrix $\mathcal{M} = (1, \ldots, 1)^{\mathrm{tr}} \in \mathrm{Mat}_{s,1}(\mathbb{R})$, and $d = 0$.*
- **A2** *Increase d by one and let $L = [t_1, \ldots, t_\ell]$ be the list of all terms of degree d in $\partial\mathcal{O}$, ordered decreasingly w.r.t. σ. If $L = \emptyset$, return the pair $(\mathcal{O}, G)$ and stop.*
- **A3** *Form the matrix $\mathcal{A} = (\mathrm{eval}(t_1), \ldots, \mathrm{eval}(t_\ell), \mathcal{M})$ and calculate a matrix $\mathcal{B}$ whose rows are an ONB of the approximate kernel $\mathrm{apker}(\mathcal{A}, \varepsilon)$ of $\mathcal{A}$.*
- **A4** *Compute the stabilized reduced row echelon form of $\mathcal{B}$ with respect to the given τ. The result is a matrix $\mathcal{C} = (c_{ij}) \in \mathrm{Mat}_{k,\ell+m}(\mathbb{R})$ such that $c_{ij} = 0$ for $j < \nu(i)$. Here $\nu(i)$ denotes the column index of the pivot element in the i^{th} row of $\mathcal{C}$.*
- **A5** *For all $j \in \{1, \ldots, \ell\}$ such that there exists an $i \in \{1, \ldots, k\}$ with $\nu(i) = j$, append the polynomial*

$$c_{ij} t_j + \sum_{j'=j+1}^{\ell} c_{ij'} t_{j'} + \sum_{j'=\ell+1}^{\ell+m} c_{ij'} u_{j'}$$

to the list G, where $u_{j'}$ is the $(j'-\ell)^{\rm th}$ element of $\mathcal{O}$.

A6 *For all $j=\ell,\ell-1,\dots,1$ such that the $j^{\rm th}$ column of $\mathcal{C}$ contains no pivot element, append the term t_j as a new first element to $\mathcal{O}$ and append the column $\operatorname{eval}(t_j)$ as a new first column to $\mathcal{M}$.*

A7 *Calculate a matrix $\mathcal{B}$ whose rows are an ONB of $\operatorname{apker}(\mathcal{M},\varepsilon)$.*

A8 *Repeat steps **A4** – **A7** until $\mathcal{B}$ is empty. Then continue with step **A2**.*

This is an algorithm which computes a pair $(\mathcal{O},G)$ such that the following properties hold for the bounds δ and η given in [5], Thm. 3.3.

(a) *The set G consists of unitary polynomials which vanish δ-approximately at the points of $\mathbb{X}$.*
(b) *The set $\mathcal{O}=\{t_1,\dots,t_\mu\}$ contains an order ideal of terms such that there is no unitary polynomial in $\langle\mathcal{O}\rangle_K$ which vanishes ε-approximately on $\mathbb{X}$.*
(c) *The set $\widetilde{G}=\{(1/\operatorname{LC}_\sigma(g))\,g\mid g\in G\}$ is an $\mathcal{O}$-border prebasis.*
(d) *The set $\widetilde{G}$ is an η-approximate border basis.*

Our main algorithm combines the techniques of this AVI-algorithm with the subideal version of the BM-algorithm presented above (see Alg. 4.2). The result is an algorithm which computes an approximate subideal border basis. This notion is defined as follows.

Definition 5.3. Let $\mathcal{O}_F=\{t_1f_{\alpha_1},\dots,t_\mu f_{\alpha_\mu}\}$ be an F-order ideal, let $\partial\mathcal{O}_F=\{b_1f_{\beta_1},\dots,b_\nu f_{\beta_\nu}\}$ be its border, and let $G=\{g_1,\dots,g_\nu\}$ be an $\mathcal{O}_F$-subideal border prebasis. Recall that this means that g_j is of the form $g_j=b_jf_{\beta_j}-\sum_{i=1}^{\mu}c_{ij}t_if_{\alpha_i}$ with $c_{ij}\in\mathbb{R}$.

For every pair (i,j) such that b_i,b_j are neighbors in $\partial\mathcal{O}_F$, i.e. such that $\beta_i=\beta_j$ and b_i,b_j are neighbors in the usual sense, we compute the normal remainder $S'_{ij}=\operatorname{NR}_{\mathcal{O}_F,G}(S_{ij})$ of the S-polynomial of g_i and g_j with respect to G. We say that G is an *ε-approximate $\mathcal{O}_F$-subideal border basis* if we have $\|S_{ij}\|<\varepsilon$ for all such pairs (i,j).

Now we are ready to formulate and proof the main result of this section.

Algorithm 5.4. (Subideal Version of the AVI-Algorithm)
Let $\mathbb{X}=\{p_1,\dots,p_s\}\subset[-1,1]^n\subset\mathbb{R}^n$ be a set of points as above, let σ be a degree compatible term ordering, and let $F=\{f_1,\dots,f_m\}\subset P\setminus\{0\}$ be a set of $\|\ \|_1$-unitary polynomials which generate an ideal $J=\langle F\rangle$. Consider the following sequence of instructions.

SA1 *Let $d=\min\{\deg(f_1),\dots,\deg(f_m)\}-1$, $\mathcal{O}_F=\emptyset$, $G=\emptyset$, and $\mathcal{M}\in\operatorname{Mat}_{s,0}(K)$.*

SA2 *Increase d by one. Let $L=[t_1f_{\alpha_1},\dots,t_\ell f_{\alpha_\ell}]$ be the list of all F-terms of degree d in $F\cup\partial\mathcal{O}_F$, with their leading terms ordered decreasingly w.r.t. σ. If then $L=\emptyset$ and $d\ge\max\{\deg(f_1),\dots,\deg(f_m)\}$, return $(\mathcal{O}_F,G)$ and stop.*

SA3 *Form the matrix $\mathcal{A}=(\operatorname{eval}(t_1f_{\alpha_1})\mid\cdots\mid\operatorname{eval}(t_\ell f_{\alpha_\ell})\mid\mathcal{M})$ and compute a matrix $\mathcal{B}$ whose rows are an ONB of the approximate kernel of $\mathcal{A}$.*

SA4 *Compute the stabilized reduced row echelon form of $\mathcal{B}$ with respect to the given τ. The result is a matrix $\mathcal{C}=(c_{ij})\in\operatorname{Mat}_{k,\ell+m}(\mathbb{R})$ such that $c_{ij}=0$ for $j<\nu(i)$. Here $\nu(i)$ denotes the column index of the pivot element in the $i^{\rm th}$ row of $\mathcal{C}$.*

SA5 *For all $j \in \{1, \ldots, \ell\}$ such that there exists an $i \in \{1, \ldots, k\}$ with $\nu(i) = j$, append the polynomial*

$$t_j f_{\alpha_j} + \sum_{j'=j+1}^{\ell} c_{ij'} t_{j'} f_{\alpha_{j'}} + \sum_{j'=\ell+1}^{\ell+m} c_{ij'} u_{j'}$$

to the list G, where $u_{j'}$ is the $(j'-\ell)^{\text{th}}$ element of $\mathcal{O}_F$.

SA6 *For all $j = \ell, \ell-1, \ldots, 1$ such that the j^{th} column of $\mathcal{C}$ contains no pivot element, append the F-term $t_j f_{\alpha_j}$ as a new first element to $\mathcal{O}_F$, append the column $\operatorname{eval}(t_j f_{\alpha_j})$ as a new first column to $\mathcal{M}$.*

SA7 *Calculate a matrix $\mathcal{B}$ whose rows are an ONB of $\operatorname{apker}(\mathcal{M}, \varepsilon)$.*

SA8 *Repeat steps **SA4** – **SA7** until $\mathcal{B}$ is empty. Then continue with step **A2**.*

This is an algorithm which computes a pair $(\mathcal{O}_F, G)$ with the following properties:

(a) *The set G consists of unitary polynomials which vanish δ-approximately at the points of $\mathbb{X}$. Here we can use $\delta = \varepsilon\sqrt{\nu} + \tau\nu(\mu+\nu)\sqrt{s}$.*

(b) *The set $\mathcal{O}_F$ contains an F-order ideal such that there is no unitary polynomial in $\langle \mathcal{O}_F \rangle_K$ which vanishes ε-approximately on $\mathbb{X}$.*

(c) *The set $\widetilde{G} = \{(1/\operatorname{LC}_\sigma(g))\, g \mid g \in G\}$ is an $\mathcal{O}_F$-subideal border prebasis.*

(d) *The set $\widetilde{G}$ is an η-approximate subideal border basis for $\eta = 2\delta + 2\nu\delta^2/\gamma\varepsilon + 2\nu\delta\sqrt{s}/\varepsilon$. Here γ denotes the smallest absolute value of the border F-term coefficient of one the polynomials g_i.*

Proof. Large parts of this proof correspond exactly to the proof of the usual AVI-algorithm (see Thm. 3.2 in [5]). Therefore we will mainly point of the additional arguments necessary to show the subideal version. The finiteness proof is identical to the finiteness proof in the subideal version of the BM-algorithm 4.2.

For the proof of (a), we can proceed exactly as in the case of the usual AVI-algorithm. There is only one point where we have to provide a further argument: the norm of the evaluation vector of an F-term is $\le \sqrt{s}$. To see this, we let $t_i f_j$ be an F-term and we write $t_i f_j = \sum_k c_k \tilde{t}_k$ with $c_k \in \mathbb{R}$ and $\tilde{t}_k \in \mathbb{T}^n$. Since f_j is $\|\ \|_1$-unitary and $\mathbb{X} \in [-1,1]^n$, we have $\|\operatorname{eval}(t_i f_j)\| \le \sum_k |c_k|\, \|\operatorname{eval}(\tilde{t}_k)\| \le \|f_j\|_1 \sqrt{s} = \sqrt{s}$.

Next we show (b). The columns of the final matrix $\mathcal{M}$ are precisely the evaluation vectors of the F-terms in $\mathcal{O}_F$. After the loop in steps **SA4** – **SA8**, we have $\operatorname{apker}(\mathcal{M}) = \{0\}$. Hence no unitary polynomial in $\langle \mathcal{O}_F \rangle_K$ has an evaluation vector which is smaller than ε. It remains to show that $\mathcal{O}_F$ is an F-order ideal. Suppose that $t_i f_j \in \mathcal{O}_F$ and that $x_k t_i f_j$ is put into $\mathcal{O}_F$. We have to prove that every F-term $\tilde{t}\, f_j$ such that $x_\ell \tilde{t} f_j = x_k t_i f_j$ is also contained in $\mathcal{O}_F$. In this case we have $t_i = x_\ell t'$ and we want to show $x_k t' f_j \in \mathcal{O}_F$. For a contradiction, suppose that $x_k t' f_j$ is the border F-term of some $g \in G$. Since the evaluation vector of $x_\ell x_k t' f_j = x_k t_i f_j$ is not larger than $\operatorname{eval}(x_k t' f_j)$, also this F-term would be detected by the loop of steps **SA4** – **SA8** as the border F-term of an element of G. This contradicts $x_k t_i f_j \in \mathcal{O}_F$.

To prove (c), it suffices to note that steps **SA2** and **SA5** make sure that the elements of G have the necessary form. Finally, claim (d) follows in exactly the same way as part (d) of [5], Thm. 3.3. □

Let us follow the steps of this algorithm in a concrete case which is a slightly perturbed version of Example 4.3.

Example 5.5. In the ring $P = \mathbb{R}[x,y,z]$ we consider the ideal $J = \langle f_1, f_2\rangle$ generated by the $\|\ \|_1$-unitary polynomials $f_1 = 0.5\,y - 0.5\,z$ and $f_2 = 0.5\,x^2 - 0.5$. Let $\sigma = \texttt{DegRevLex}$, let $\varepsilon = 0.03$, and let $\tau = 0.001$. We want to compute an approximate subideal border basis vanishing approximately at the points of $\mathbb{X} = \{(1,1,1),\ (0,1,1),\ (1,1,0),\ (1,0,0.98),\ (0.98,0,1)$.

Notice that the first point of $\mathbb{X}$ is contained in $\mathcal{Z}(f_1,f_2)$ and that the last two points of $\mathbb{X}$ differ by $\le \varepsilon$ from one point $(1,0,1)$. Hence the approximate subideal border basis should correspond to *three* points outside $\mathcal{Z}(J)$, and therefore we should expect to get an F-order ideal consisting of three F-terms. We follow the steps of the subideal version of the AVI-algorithm 5.4.

SA2 Let $d = 1$ and $L = [0.5\,y - 0.5\,z]$.
SA3 We compute $\mathcal{A} = (0,0,0.5,-0.49,-0.51)^{\mathrm{tr}}$ and $\mathcal{B} = (0)$. (Thus $\mathcal{C} = \mathcal{B}$.)
SA6 Let $\mathcal{O} = \{f_1\}$ and $\mathcal{M} = (0,0,0.5,-0.49,-0.5)^{\mathrm{tr}}$.
SA2 Let $d = 2$ and $L = [f_2,\ xf_1,\ yf_1,\ zf_1]$.

SA3 We compute $\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0.5 \\ 0 & -0.49 & 0 & -0.4802 & -0.49 \\ -0.0198 & -0.49 & 0 & -0.5 & -0.5 \end{pmatrix}$ and

$$\mathcal{B} = \operatorname{apker}(\mathcal{A},\varepsilon) = \begin{pmatrix} 0.0004 & 0.6755 & -0.5089 & -0.5068 & -0.1667 \\ 0 & -0.3812 & -0.3735 & -0.3812 & 0.7548 \end{pmatrix}.$$

SA4 The stabilized reduced row echelon form of $\mathcal{B}$ is

$$\mathcal{C} = \begin{pmatrix} 0 & 0.3812 & 0.3735 & 0.3812 & -0.7548 \\ 0 & 0 & 0.5754 & 0.5811 & -0.5754 \end{pmatrix}.$$

SA5 We get $G = \{g_1, g_2\}$ with $g_1 = 0.3812\,xf_1 + 0.3735\,yf_1 + 0.3812\,zf_1 - 0.7548f_1$ and $g_2 = 0.5754\,yf_1 + 0.5811\,zf_1 - 0.5754f_1$.

SA6 We find $\mathcal{O} = \{f_2,\ zf_1,\ f_1\}$ and $\mathcal{M} = \begin{pmatrix} 0 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4802 & -0.5 \\ 0 & 0 & 0.5 & -0.49 & -0.5 \end{pmatrix}^{\mathrm{tr}}$.

SA2 Now let $d = 3$ and $L = [xf_2,\ yf_2,\ zf_2, xzf_1,\ yzf_1,\ z^2f_1]$.

SA3 $\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & -0.5 & 0 & 0 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & -0.48 & 0 & -0.47 & 0 & -0.48 & -0.49 \\ -0.02 & 0 & -0.02 & -0.49 & 0 & -0.5 & -0.02 & -0.5 & -0.5 \end{pmatrix}$

and $\mathcal{B},\mathcal{C}$ are matrices of rank 6 which yield six further approximate subideal border basis elements.
SA5 We obtain $G = \{g_1,\dots,g_8\}$ with $g_3 = xf_2 - 0.02\,zf_1$, $g_4 = 0.71\,yf_2 - 0.71\,f_2 + 0.01\,zf_1$, $g_5 = 0.71\,zf_2 - 0.71\,f_2$, $g_6 = 0.71\,xzf_1 - 0.7\,zf_1$, $g_7 = yzf_1$, and $g_8 = 0.71\,z^2f_1 - 0.7\,zf_1$.
SA5 Since there is no new non-pivot row index, $\mathcal{O}_F$ and $\mathcal{M}$ are not changed.
SA2 In degree $d = 4$ we find $L = \emptyset$ and the algorithm stops.

Hence the result is the F-order ideal $\mathcal{O} = \{x^2 - 1,\ z(y-z),\ y - z\}$ and the approximate $\mathcal{O}_F$-subideal border basis $G = \{g_1,\dots,g_8\}$. This confirms that there are three approximate zeros of G outside the two lines $\mathcal{Z}(f_1,f_2)$.

6. An Industrial Application

In this section we apply the subideal version of the AVI-algorithm to an actual industrial problem which has been studied in the Algebraic Oil Research Project (see [2]). Viewed from a more general perspective, this application shows how one can carry out the suggestion made in the introduction, namely to use the subideal version of the AVI-algorithm to introduce knowledge about the nature of a physical system into the modeling process.

Suppose that a multi-zone well consists of two zones A and B. During so-called *commingled production*, the two zones are interacting and influence each other. We have at our disposal time series of measured data such as pressures, temperatures, total production and valve positions. Moreover, during so-called *test phases* we can obtain time series of these data when only one of the two zones is producing. The following figure gives a schematic representation of the physical system and the measured variables.

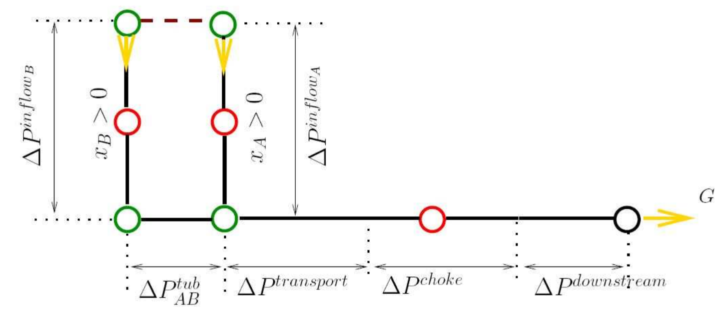

Figure 1. Schematic representation of a two-zone well

The measured total production does not equal the sum of the individual productions calculated from the test data. The *production allocation problem* is to determine the contributions of the two zones to the total production when they are producing together. Here the *contributions* c_A, c_B of the zones are defined to be the part of the total production p_{AB} passing through the corresponding down-hole valves. Therefore we have $p_{AB} = c_A + c_B$, but there is no way of measuring c_A and c_B directly. In this sense the production allocation problem is to determine the contributions c_A, c_B from the measured data.

Let the indeterminate x_A represent the valve position of zone A and x_B the valve position of zone B. Here $x_i = 0$ means that the valve is closed and $x_i = 1$ represents a fully opened valve position. Clearly, if valve A is closed, i.e. for points in the zero set $\mathcal{Z}(\langle x_A \rangle)$, there is no contribution from zone A, and likewise for B. By Hilbert's Nullstellensatz, this means that the polynomial p_A modeling the production of zone A should be computed by using the subideal version of the AVI-algorithm with $J = \langle x_A \rangle$. Similarly, we want to force $p_B \in \langle x_B \rangle$.

Now we model the total production p_{AB} in the following way. We write $p_{AB} = p_A + p_B + q_{AB}$ where q_{AB} is a polynomial which measures the interaction of the two zones. To compute q_{AB}, we write it in the form

$$q_{AB} = f_A \cdot (x_B \cdot p_A) + f_B \cdot (x_A \cdot p_B)$$

Notice that such a decomposition can be computed via the subideal version of the AVI-algorithm by applying it to the ideal $J = \langle x_B p_A,\, x_A p_B \rangle$. The result will be a representation $p_{AB} = p_A + p_B + f_A x_B p_A + f_B x_A p_B$. Here we observe that $x_A = 0$ implies $p_{AB} = p_B$ because $p_A \in \langle x_A \rangle$. Analogously, we see that $x_B = 0$ implies $p_{AB} = p_A$, in accordance with the physical situation.

The endresult of these computations is that the contributions of the two zones during commingled production can be computed from the equalities $c_A = (1 + f_A x_B) p_A$ and $c_B = (1 + f_B x_A) p_B$. At the same time we gain a detailed insight into the nature of the interactions by examining the structure of the polynomials f_A, f_B.

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