Chapter 1
From Oil Fields to Hilbert Schemes

Martin Kreuzer, Hennie Poulisse, and Lorenzo Robbiano

Abstract New techniques for dealing with problems of numerical stability in computations involving multivariate polynomials allow a new approach to real world problems. Using a modeling problem for the optimization of oil production as a motivation, we present several recent developments involving border bases of polynomial ideals. After recalling the foundations of border basis theory in the exact case, we present a number of approximate techniques such as the eigenvalue method for polynomial system solving, the AVI algorithm for computing approximate border bases, and the SOI algorithm for computing stable order ideals. To get a deeper understanding for the algebra underlying this *approximate world*, we present recent advances concerning border basis and Gröbner basis schemes. They are open subschemes of Hilbert schemes and parametrize flat families of border bases and Gröbner bases. For the reader it will be a long, tortuous, sometimes dangerous, and hopefully fascinating journey from oil fields to Hilbert schemes.

Key words: oil field, polynomial system solving, eigenvalue method, Buchberger-Möller algorithm, border basis, approximate algorithm, border basis scheme, Gröbner basis scheme, Hilbert scheme

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Introduction

Why did the chicken cross the road?
To boldly go where no chicken has gone before.
(James Tiberius Kirk)

A Bridge Between Two Worlds. Oil fields and Hilbert schemes are connected to very different types of ingredients for algorithmic and algebraic manipulation: continuous and discrete data. This apparent dichotomy occurs already in a single polynomial over the real number field. It consists of a discrete part, the support, and a continuous part, the set of its coefficients. The support is well understood and the source of a large amount of literature in classical algebra. On the other hand, if the coefficients are not exact real numbers but *approximate data*, the very notion of a polynomial and all algebraic structures classically derived from it (such as ideals, free resolutions, Hilbert functions, etc.) tend to acquire a blurred meaning.

An easy example is the following. Consider three distinct non-aligned points in the affine plane over the reals. First of all, if the coordinates are not exact, it is not even clear what we mean by "non-aligned"; a better description might be "far from aligned". The vanishing ideal of the three points is generated by three quadratic polynomials. However, if we change some of the coefficients of these polynomials by a small amount, almost surely we get the unit ideal, since the first two conics still intersect in four points, but the third will almost certainly miss all of them.

How can we cope with this situation? And why should we? The first, easy answer is that approximate coefficients are virtually inevitable when we deal with *real world problems*. In this paper we concentrate on a specific problem where vectors with approximate components encode measurements of physical quantities taken in an **oil field**. Based on actual industrial problems in the field of oil production, we want to popularize the idea that good models of many physical phenomena can be constructed using a *bottom-up* process. The heart of this method is to derive mathematical models by interpolating measured values on a finite set of points. This task can be solved if we know the vanishing ideal of the point set and a suitable vector space basis of its coordinate ring.

This leads us to the next question. Given a zero-dimensional ideal I in a polynomial ring over the reals, if we assume that the coefficients of the generating polynomials are inexact, is it still an ideal? What is the best way of describing this situation? The fact that Gröbner bases are not suitable for computations with inexact data has long been well-known to numerical analysts (see [30]). This is due to the rigid structure imposed by term orderings. Other objects, called *border bases*, behave better. They have emerged as good candidates to complement, and in many cases substitute for, Gröbner bases (see [17], [21], [22], [26], [29]). But possibly the most important breakthrough is the recent discovery of a link between border bases and **Hilbert schemes**. We believe that it may provide a solid mathematical foundation for this new emerging field which tries to combine approximate methods from numerical analysis with exact methods from commutative algebra and algebraic geometry.

You got to be careful if you don't know where you're going
because you might not get there.
(Yogi Berra)

Our Itinerary. In the first part of the introduction we have already suggested the existence of an unexpected bridge between oil fields and Hilbert schemes. Let us now be more specific about the content of the paper and indicate how it tries to build that bridge. Section 1 provides an introduction to one of the main problems arising in oil fields, namely the control of the production. Since we assume that our typical reader is not an expert geologist, we provide some background about the physical nature of an oil reservoir, illustrate the main production problem, and describe a new mathematical approach to solve it. We call it "new", since in our opinion it is very different from the standard view on how to use mathematical models in such a context.

Border bases, the main technical tool we use later, are described in Section 2. This material is mainly taken from [21], Section 6.4 and [17]. We describe the definition and the main properties of border bases and compare them to Gröbner bases using suitable examples. Several important results about border bases are described, in particular their characterization via the commutativity of the formal multiplication matrices due to B. Mourrain (see [26]). A brief excursion is taken into the realm of syzygies, their relation to the border web, and their importance in another fundamental characterization of border bases based on the work of H. Stetter (see [30]).

A useful aspect of border basis theory is that we try to specify a "nice" vector space basis of the quotient ring $\mathbb{R}[x_1,\ldots,x_n]/I$. This sort of basis plays a fundamental role in the problem of solving polynomial systems. Notwithstanding the fact that solving polynomial systems is not a main topic in our presentation, we decided to use Section 3 to give a description of a technique which comes from numerical analysis and uses linear algebra methods, in particular eigenvalues and eigenvectors (see [4], [5], and [9]). The importance of a special kind of matrices, called non-derogatory matrices, is illustrated by Example 1.3.9 and also used in [19] in the context of border basis theory.

Sections 4 and 5 are the computational heart of the paper. They describe two somehow complementary algorithmic approaches to the problem of computing the

"approximate vanishing ideal" of a finite set of approximate (empirical) points and a basis of the corresponding quotient ring. In particular, the first part of Section 4 deals with the AVI algorithm and is based on the presentation in [14]. The AVI algorithm makes extensive use of the singular value decomposition (SVD) described in Subsection 4.A and of the stable reduced row echelon form explained in Subsection 4.B. Its main outputs are an order ideal of monomials $\mathcal{O}$ and an approximate $\mathcal{O}$-border basis, a concept introduced in Subsection 4.C. The AVI algorithm is then applied in Subsection 4.D to the concrete construction of polynomial models describing the production of a two-zone oil well.

Section 5 deals with the SOI algorithm which treats the following problem: given a finite set of points $\mathbb{X}$ whose coordinates are given with limited precision, find, if there exists one, an order ideal $\mathcal{O}$ such that the residue classes of its elements form a stable basis of the quotient ring $P/\mathcal{I}(\mathbb{X})$ where $P = \mathbb{R}[x_1,\ldots,x_n]$ and $\mathcal{I}(\mathbb{X})$ is the vanishing ideal of $\mathbb{X}$. Here stable means that the residue classes of the elements in $\mathcal{O}$ form a basis of the quotient ring for every small perturbation of the set $\mathbb{X}$. This section summarizes the results of [2]. In Subsection 5.B we describe several easy, but illustrative examples and compare the behaviour of the SOI and the AVI algorithm in these cases. The topic studied in Sections 4 and 5 is an active area of research, and several further approaches have been suggested (see for instance [10] and [25]).

Having done all the *dirty* work (oil fields are not places to be dressed formally), it is time to leave the sedimentary rocks and to look at the problems concerning approximate data from a more general perspective. Polynomials with empirical coefficients can be viewed as *families of polynomials*. So, the next question is whether we can describe families of polynomial ideals algebraically. The answer is yes! The possibility of parametrizing families of schemes by one big scheme is a remarkable feature of algebraic geometry. Hilbert schemes are the most widely known instance of this phenomenon, and consequently they have been studied thoroughly. Moreover, the Hilbert scheme of all zero-dimensional ideals in P of colength s can be covered by affine open subschemes which parametrize all subschemes $\operatorname{Spec}(P/I)$ of the affine space $\mathbb{A}^n_K$ with the property that P/I has a fixed vector space basis. It is interesting to note that the construction of such subschemes is performed using border bases (see for instance [15], [16], and [24]). Also Gröbner bases can be used, since they provide tools for constructing suitable stratifications of Hilbert schemes.

Section 6 is devoted to the explanation of these ideas. Its main sources are the two papers [22] and [28]. In Subsection 6.A we start with an informal explanation of two examples (see Examples 1.6.1 and 1.6.2) which are very easy but nevertheless suitable to illustrate the topic. Then we move to Subsection 6.B where we introduce border basis schemes and their associated border basis families. We show the difficulties of generalizing one of the fundamental tools of Gröbner basis theory to the border basis setting, namely the flat deformation to the leading term ideal. Indeed, the problem is only partially solved and still open in general. The final part of the subsection contains Example 1.6.14 where explicit defining equations are given for one particular border basis scheme, and the connection to the approximate border bases of Section 4 is made.

The final Subsection 6.C is devoted to Gröbner basis schemes and summarizes the presentation in [28]. It is shown that Gröbner basis schemes and their associated universal families can be viewed as weighted projective schemes (see Theorem 1.6.19), a fact that constitutes a remarkable difference between Gröbner and border basis schemes. A comparison between the two types of schemes is given by Theorem 1.6.20 and Corollary 1.6.21, and their equality is examined in Proposition 1.6.24. Throughout the section we highlight the connection between border basis schemes, Gröbner basis schemes, and Hilbert schemes.

At that point the journey from oil fields to Hilbert schemes is over. To get you started with this itinerary, let us point out that, unless specifically stated otherwise, our notation follows the two books [20] and [21]. The algorithms we discuss have been implemented in the computer algebra system CoCoA(see [8]) and in the ApCoCoA library (see [3]).

1.1 A Problem Arising in Industrial Mathematics

Are oil fields commutative?
Are they infinite?
What is their characteristic?
Are they stable?
What are their bases?
(from "The Book of Mathematical Geology")

1.1.A. Oil Fields, Gas Fields and Drilling Wells. Research in relation to oil reservoirs faces many times the same kind of difficulty: the true physical state of an intact, working reservoir cannot be observed. Neither in an experiment of thought, for instance a simulation, nor in a physical experiment using a piece of source rock in a laboratory, the reservoir circumstances can be imitated exactly. This means that the physical laws, i.e. the relations between the physical quantities, are not known under actual reservoir circumstances.

To shed some additional light upon this problem, let us have a brief look at oil field formation and exploitation. The uppermost crust of the earth in oil and gas-containing areas is composed of sedimentary rock layers. Since the densities of oil and gas are smaller than the density of water, buoyancy forces them to flow upward through small pores in the reservoir rock. When they encounter a *trap*, e.g. a dome or an anticline, they are stopped and concentrated according to their density: the gas is on top and forms the free *gas cap*, the oil goes in the middle, and the (salt) water is at the bottom. To complete the trap, a *caprock*, that is a seal which does not allow fluids to flow through it, must overlie the reservoir rock.

Early drillings had some success because many subsurface traps were leaking. Only by the early 1900s it became known that traps could be located by mapping the rock layers and drilling an exploration well to find a new reservoir. If commercial amounts of oil and gas turn out to be present, a long piece of steel pipe (called the

production tubing) is lowered into the bore hole and connected to the production facilities.

In a gas well, gas flows to the surface by itself. There exist some oil wells, early in the development of an oil field, in which the oil has enough pressure to flow up the surface. Most oil wells, however, do not have enough pressure and a method called *artificial lift* may then be used. This means that gas is injected into the production tubing of the well. The injected gas mixes with the oil and makes it lighter, thereby reducing the back pressure of the reservoir. On the surface the fluids are transported through long pieces of tubing to a large vessel called *separator* where the three physical phases – oil, water and gas – are separated.

During the exploitation of a reservoir, the pressure of the fluid still in the reservoir drops. This decrease of the reservoir pressure over time is depicted by the *decline curve*. The shape of the decline curve and the total volume of fluid that can be produced from a reservoir (which is called the *ultimate recovery*) depend on the *reservoir drive*, the natural energy that pushes the oil or the gas through the subsurface and into the inflow region of the well. The ultimate recovery of gas from a gas reservoir is often about 80% of the gas in the reservoir. Oil reservoirs are far more variable and less efficient: on average, the ultimate recovery is only 30%. This leaves 70% of the oil remaining in the pressure depleted reservoir which cannot be produced anymore.

Thus, on the most abstract level, the problem we want to address is how to increase the ultimate recovery of an oil reservoir.

1.1.B. Production from Multi-Zone Wells. A well may produce from different parts, called *pockets* or *zones*, of an oil reservoir. The total production of such a well consists of contributions from the different zones. The separate contributions can be controlled by valves, called the *down-hole valves*, which determine the production volume flowing into the well tubing at the locations of the different zones. For such a *multi-zone well*, there may be interactions between the zones in the reservoir. Most certainly, the different contributions will interact with each other when they meet in the common production tubing of the multi-zone well. This situation is called *commingled* production.

In this paper we consider a multi-zone well consisting of **two** producing and interacting zones. Like in a single oil well, the common production flows to the bulk separator where the different phases are separated and the production rates of the separated phases are measured. Besides the phase productions, measurements like pressures, temperatures and injected "lift-gas" are collected; down-hole valves positions are also recorded. A typical set of production variables for a such multi-zone well is:

1. the opening of the valve through which the oil from the first zone is entering the multi-zone well; the opening of the valve is measured in percentages: 0% means that the valve is closed; 100% means that the valve is completely open;
2. the opening of the valve through which the oil from the second zone is entering the multi-zone well;

3. the pressure difference over the down-hole valve of the second zone which is a measure for the inflow from the reservoir into the well at the valve position; if the valve is closed we assume this value to be zero;
4. the pressure difference over the down-hole valve of the first zone when the valve in that zone is open; if the valve is closed we assume this value to be zero;
5. the volume of gas produced simultaneously with the oil;
6. the pressure difference between the inflow locations in the production tubing;
7. the pressure difference which drives the oil through the transportation tubing.

One might be tempted to think that the total oil production of a multi-zone well is the sum of the productions of each zone when producing separately. This is in any case the current state of the art, where the total production is regressed against the separate productions, that is the total production is written as a linear combination of the separate productions. The coefficients in this linear sum are called *reconciliation factors*. The oil produced by one of the zones may push back the oil which tries to flow into the well at the other zone. Likewise, the gas which is produced simultaneously with the oil may have stimulating or inhibiting effects on the inflow of the oil with respect to the situation of single zone productions. With reference to the remarks above, this behavior does not sound very linear. Indeed, in Section 4.D we will use our algebraic approach in a two-zone well example to demonstrate that the total production is not a linear combination of the separate productions. We believe that the reason of the (usually) low ultimate recovery of a multi-zone well is due to the fact that the interactions among the different producing zones are unknown.

This leads us to a first concretization of the problem we want to study: find a model for the total production of an oil well which takes the interactions into account and describes the behavior correctly on longer time scales.

1.1.C. Algebraization of the Production Problem. Before plunging into the creation of an algebraic setting for the described production problem, let us spend a few words on why we believe that approximate computational algebra is an appropriate method to deal with it.

The available data correspond to a finite set of points $\mathbb{X}$ in $\mathbb{R}^n$. Their coordinates are *noisy* measurements of physical quantities associated with the well: pressures, oil and gas production, valve positions, etc. These points represent the behavior of the well under various production conditions. The combination of the contribution of the individual zones to the total production is a *sum* which has to be corrected by taking into account the effect of the interactions. As in many other situations (for instance, in statistics), the interactions are related to *products* of the collected data series. Many of the known physical laws and model equations are of a polynomial nature. And even if they are not, some elementary insights into the system (e.g. that the result depends exponentially on a certain data series) allow us to prepare the data series appropriately (e.g. by computing their logarithms). Consequently, the starting point for us is the polynomial ring $P = \mathbb{R}[x_1, \ldots, x_n]$.

In the following we will deal with the case of a two-zone well. The production situation is depicted schematically in Figure 1.1. The notation ΔP refers to pressure differences.

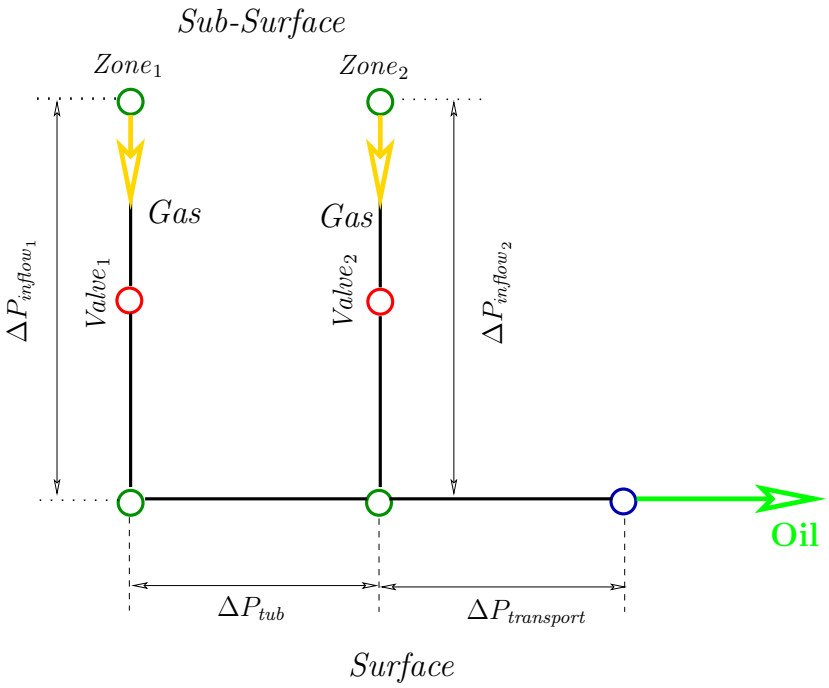


Fig. 1.1 Schematic representation of a two-zone well.

The valves indicated in this figure are used to influence the inflow of the fluids at the two locations into the production tubing of the well. If a valve is closed, there is no inflow from the reservoir at the location of the valve. If the valve is open, the inflow depends on the valve opening and the interactions with the fluids which enter the well through the other inflow opening. In particular, a valve in open position does not imply that there is inflow from the reservoir into the well at its location.

Next we try to formulate the problems associated with this production system more explicitly. Notice that the reservoir is a very special physical system in that it is not possible to check "how it works" using a computer simulation experiment or a physical model laboratory experiment. Traditional modeling techniques assume that equations which describe the flow of the fluids through the reservoir are available. Their limited success is in our view due to the fact that there is no proper representation of the interactions occurring in the production situation. Without these, actions taken to influence the production may have devastating consequences in that the "wrong" effects are stimulated. It is fair to state that the existing low ultimate recovery rates are to a large extent caused by the fact that the interactions in production units have not been acknowledged properly.

As a starting point, let us formulate the production problem in intuitive rather than in precise mathematical terms.

Problem 1. Assume that no a priori model is available to describe the production of the two-zone well of Figure 1.1 in terms of measurable physical quantities which determine the production. Find an algebraic model of the production in terms of the determining, measurable physical quantities which specifically models the interactions occurring in this production unit.

Now let us phrase this problem using the polynomial ring $P = \mathbb{R}[x_1, \ldots, x_n]$. The first step is to associate the indeterminates x_i with physical quantities in the production problem in the sense that when the indeterminate x_i is evaluated at the points of $\mathbb{X}$, the evaluations are the measurements of the physical quantity associated to x_i. In the sequel we use $n = 5$ and the following associations, where the physical quantities are the ones referenced in Figure 1.1.

$$
\begin{aligned}
x_1 &: \Delta P_{inflow_1} \\
x_2 &: \Delta P_{inflow_2} \\
x_3 &: \textit{Gas production} \\
x_4 &: \Delta P_{tub} \\
x_5 &: \Delta P_{transport}
\end{aligned}
$$

Table 1.1 Physical interpretation of the indeterminates.

Note that we have not listed an indeterminate associated to the oil production. The explanation for this is that the physical quantities listed in the above table may all be interpreted as *driving forces* for the oil production. For the pressure differences ΔP this is clear. But it holds also for the gas production. When a large amount of gas is produced in the deeper parts of the reservoir, it disperses in the fluid mixture, makes it lighter, and in this way stimulates oil production through this lifting process. Thus the physical quantities listed in the above table may all be viewed as the *causing* quantities, or *inputs*, and the oil production is their *effect*, or *output*. So, basically we make the following crucial assumption.

Assumption. *There exists a causal relationship between the production and the driving forces. Using suitable inputs, this causal relationship is of polynomial nature.*

Denoting the production by f, the algebraic translation of the causal relationship assumption is $f \in \mathbb{R}[x_1, \ldots, x_5]$ where the indeterminates x_i are labeled as in the above table. That is, the production is not associated with an indeterminate, but with a polynomial, and the production measurements are the evaluations of this polynomial over the set $\mathbb{X}$. Hence the problem statement 1 can be reformulated as follows.

Problem 2. Find the polynomial $f \in \mathbb{R}[x_1, \ldots, x_5]$, using only the evaluations $\mathbb{X}$ of the quantities x_i and the evaluations of f!

The information registered in the set $\mathbb{X}$ refers to the situation where at most one of the valves is closed. The only possible inflows from the reservoir into the production tubing of the two-zone well are at the location of Zone 1, or of Zone 2, or both. Moreover, in all three situations data have been collected at different valve openings. Furthermore, in order for the data in $\mathbb{X}$ to deserve the qualification *driving forces*, some pre-processing has been applied: with reference to Figure 1.1, if $valve_1$ is closed, it may very well be that the pressure difference ΔP_{inflow_1} is not zero, but it does not have the meaning of a driving force over the valve opening because there is no flow over the valve. Hence in the data set $\mathbb{X}$, we set ΔP_{inflow_1} to zero for this situation. Of course, we do the same for $valve_2$ with respect to ΔP_{inflow_2}. Finally, if the valve associated with the deepest zone $valve_1$ is closed, there is no transport of fluids in the lowest part of the production tubing of the well. That is, for ΔP_{tub} really to have the significance of a driving force, it is set to zero if $valve_1$ is closed.

Notice also that all data are based on measurements, i.e. they may contain measurement errors. Consequently, we can only expect that the desired polynomial f vanishes *approximately* at the points of $\mathbb{X}$. In Section 4 we will return to this instance of the production problem and solve it with the methods we are going to present.

1.2 Border Bases

Ideally, inside the border
there is order.
(Three anonymous authors)

1.2.A. Motivation and Definition. The problems considered in the previous section lead us to study zero-dimensional ideals in $P = K[x_1, \ldots, x_n]$ where K is a field. The two most common ways to describe such an ideal I are by either providing a special system of generators (for instance, a Gröbner basis) of I or by finding a vector space basis $\mathcal{O}$ of P/I and the matrices of the multiplications by the indeterminates with respect to $\mathcal{O}$. One possibility to follow the second approach is to use $\mathcal{O} = \mathbb{T}^n \setminus \mathrm{LT}_\sigma(I)$, the complement of a leading term ideal of I. By Macaulay's Basis Theorem, such a set $\mathcal{O}$ is a K-basis of P/I. Are there other suitable sets $\mathcal{O}$?

A natural choice is to look for sets of terms. We need to fix how a term b_j in the *border* $\partial\mathcal{O} = (x_1\mathcal{O} \cup \cdots \cup x_n\mathcal{O}) \setminus \mathcal{O}$ of $\mathcal{O}$ is rewritten as a linear combination of the terms in $\mathcal{O}$. Thus, for every $b_j \in \partial\mathcal{O}$, a polynomial of the form

$$g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$$

with $c_{ij} \in K$ and $t_i \in \mathcal{O}$ should be contained in I. Moreover, we would not like that $x_k g_j \in I$. Hence we want $x_k b_j \notin \mathcal{O}$. Therefore the set $\mathbb{T}^n \setminus \mathcal{O}$ should be a monoideal. Consequently, $\mathcal{O}$ should be an *order ideal*, that is it should be closed under forming divisors. Let us formulate precise definitions.

Definition 1.2.1. Let $\mathcal{O}$ be a finite set of terms in $\mathbb{T}^n$.

a) The set $\mathcal{O}$ is called an **order ideal** if $t \in \mathcal{O}$ and $t' \mid t$ implies $t' \in \mathcal{O}$.
b) Let $\mathcal{O}$ be an order ideal. The set $\partial\mathcal{O} = (x_1\mathcal{O} \cup \cdots \cup x_n\mathcal{O}) \setminus \mathcal{O}$ is called the **border** of $\mathcal{O}$.
c) Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal and $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ its border. A set of polynomials $\{g_1, \ldots, g_\nu\} \subset I$ of the form

$$g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$$

with $c_{ij} \in K$ and $t_i \in \mathcal{O}$ is called an **$\mathcal{O}$-border prebasis** of I.
d) An $\mathcal{O}$-border prebasis of I is called an **$\mathcal{O}$-border basis** of I if the residue classes of the terms in $\mathcal{O}$ are a K-vector space basis of P/I.

The following example will be used frequently throughout this paper.

Example 1.2.2. In the ring $P = \mathbb{R}[x,y]$, consider the ideal $I = (f_1, f_2)$ where

$$\begin{aligned} f_1 &= \tfrac{1}{4}x^2 + y^2 - 1 \\ f_2 &= x^2 + \tfrac{1}{4}y^2 - 1 \end{aligned}$$

The zero set of I in $\mathbb{A}^2(\mathbb{R})$ consists of the four points $\mathbb{X} = \{(\pm\sqrt{0.8}, \pm\sqrt{0.8})\}$. This setting is illustrated in Figure 1.2.

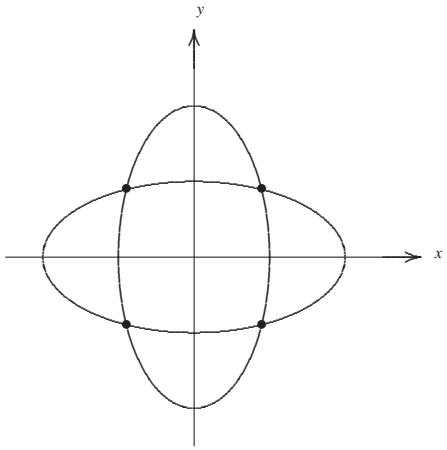


Fig. 1.2 Two ellipses intersecting in four points.

We use $\sigma = \texttt{DegRevLex}$ and compute $\mathrm{LT}_\sigma(I) = (x^2, y^2)$. Thus the order ideal $\mathcal{O} = \{1, x, y, xy\}$ represents a basis of P/I. Its border is $\partial\mathcal{O} = \{x^2, x^2y, xy^2, y^2\}$. The following figure illustrates the order ideal $\mathcal{O}$ and its border.

An $\mathcal{O}$-border basis of I is given by $G = \{g_1, g_2, g_3, g_4\}$ where

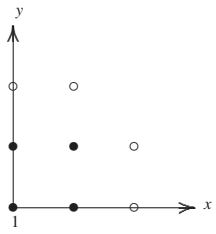


Fig. 1.3 An order ideal and its border.

$$\begin{aligned} g_1 &= x^2 - 0.8 \\ g_2 &= x^2 y - 0.8\, y \\ g_3 &= x y^2 - 0.8\, x \\ g_4 &= y^2 - 0.8 \end{aligned}$$

Let us see what happens if we disturb this example slightly.

Example 1.2.3. Again we use $P = \mathbb{R}[x,y]$, but now we consider $\tilde{I} = (\tilde{f}_1, \tilde{f}_2)$ where

$$\begin{aligned} \tilde{f}_1 &= 0.25\, x^2 + y^2 + 0.01\, xy - 1 \\ \tilde{f}_2 &= x^2 + 0.25\, y^2 + 0.01\, xy - 1 \end{aligned}$$

Its zero set consists of four perturbed points $\widetilde{\mathbb{X}}$ close to those in $\mathbb{X}$, as illustrated in Figure 1.4.

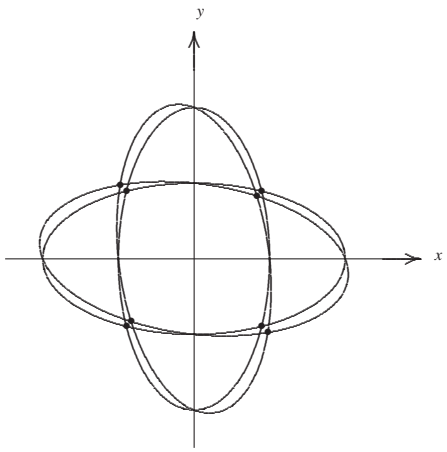


Fig. 1.4 Two slightly moved ellipses and their points of intersection.

The ideal $\tilde{I} = (\tilde{f}_1, \tilde{f}_2)$ has the reduced σ-Gröbner basis

$$\{x^2-y^2, xy+125y^2-100, y^3-\tfrac{25}{3906}x+\tfrac{3125}{3906}y\}$$

Moreover, we have $\mathrm{LT}_\sigma(\tilde{I}) = (x^2, xy, y^3)$ and $\mathbb{T}^2 \setminus \mathrm{LT}_\sigma\{\tilde{I}\} = \{1, x, y, y^2\}$.

A *small* change in the coefficients of f_1 and f_2 has led to a *big* change in the Gröbner basis of $(\tilde{f}_1, \tilde{f}_2)$ and in the associated vector space basis of $\mathbb{R}[x,y]/(\tilde{f}_1, \tilde{f}_2)$, although the zeros of the ideal have not changed much. Numerical analysts call this kind of unstable behavior a *representation singularity*.

However, also the ideal $\tilde{I}$ has a a border basis with respect to $\mathscr{O} = \{1, x, y, xy\}$. Recall that the border of $\mathscr{O}$ is $\partial\mathscr{O} = \{x^2, x^2y, xy^2, y^2\}$.

The $\mathscr{O}$-border basis of $\tilde{I}$ is $\tilde{G} = \{\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4\}$ where

$$\begin{aligned}
\tilde{g}_1 &= x^2 + 0.008xy - 0.8 \\
\tilde{g}_2 &= x^2y + \tfrac{25}{3906}x - \tfrac{3125}{3906}y \\
\tilde{g}_3 &= xy^2 - \tfrac{3125}{3906}x + \tfrac{25}{3906}y \\
\tilde{g}_4 &= y^2 + 0.008xy - 0.8\}
\end{aligned}$$

When we vary the coefficients of xy in the two generators from zero to 0.01, we can see that one border bases changes continuously into the other. Thus the border basis behaves numerically stable under small perturbations of the coefficient of xy.

1.2.B. Characterizations of border bases. In the sequel, we use the following notation: let $\mathscr{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal in $\mathbb{T}^n$, let $\partial\mathscr{O} = \{b_1, \ldots, b_\nu\}$ be the border of $\mathscr{O}$, let $G = \{g_1, \ldots, g_\nu\}$ be an $\mathscr{O}$-border prebasis, where $g_j = b_j - \sum_{i=1}^{\mu} c_{ij}t_i$ with $c_{ij} \in K$, and let $I = (g_1, \ldots, g_\nu)$ be the ideal generated by G.

The following remark collects some elementary properties of border bases.

Remark 1.2.4. Let $J \subseteq P$ be a zero-dimensional ideal.

a) The ideal J need not have an $\mathscr{O}$-border basis, even if its colength is μ. But if it does, its $\mathscr{O}$-border basis is uniquely determined.
b) If $\mathscr{O}$ is of the form $\mathbb{T}^n \setminus \mathrm{LT}_\sigma(J)$ for some term ordering σ, then J has an $\mathscr{O}$-border basis. It contains the reduced σ-Gröbner basis of J.
c) There exists a **Division Algorithm** for border prebases (see [21], 6.4.11).

The following characterizations of border bases can be shown in analogy to the corresponding results for Gröbner bases (see [21], 6.4.23 and 6.4.28). For a term $t \in \mathbb{T}^n$, its **$\mathscr{O}$-index** $\mathrm{ind}_\mathscr{O}(t)$ is the smallest natural number k such that $t = t_1t_2$ with $t_1 \in \mathscr{O}$ and $t_2 \in \mathbb{T}^n_k$.

Proposition 1.2.5. *In the above setting, the set G is an $\mathscr{O}$-border basis of I if and only if one of the following equivalent conditions is satisfied.*

a) For every $f \in I \setminus \{0\}$, there are $f_1, \ldots, f_\nu \in P$ such that $f = f_1g_1 + \cdots + f_\nu g_\nu$ and $\deg(f_i) \leq \mathrm{ind}_\mathscr{O}(f) - 1$ whenever $f_ig_i \neq 0$.
b) For every $f \in I \setminus \{0\}$, there are $f_1, \ldots, f_\nu \in P$ such that $f = f_1g_1 + \cdots + f_\nu g_\nu$ and $\max\{\deg(f_i) \mid i \in \{1, \ldots, \nu\}, f_ig_i \neq 0\} = \mathrm{ind}_\mathscr{O}(f) - 1$.

Proposition 1.2.6. *In the above setting, the set G is an $\mathcal{O}$-border basis of I if and only if the rewrite relation $\xrightarrow{G}$ associated to G is confluent.*

As we mentioned above, the vector space basis $\mathcal{O}$ of P/I can be used to describe the K-algebra structure of P/I via the multiplication matrices of the multiplication maps by the indeterminates. In addition, these multiplication maps can be used to characterize border bases, as the next theorem shows.

Definition 1.2.7. For $r \in \{1,\dots,n\}$, we define the r-th **formal multiplication matrix** $\mathcal{A}_r$ as follows:

Multiply $t_i \in \mathcal{O}$ by x_r. If $x_r t_i = b_j$ is in the border of $\mathcal{O}$, rewrite it using the prebasis polynomial $g_j = b_j - \sum_{k=1}^{\mu} c_{kj} t_k$ and put $(c_{1j},\dots,c_{\mu j})$ into the i-th column of $\mathcal{A}_r$. But if $x_r t_i = t_j$ then put the j-th unit vector into the i-th column of $\mathcal{A}_r$.

Clearly, if G is a border basis and $\mathcal{A}_1,\dots,\mathcal{A}_n$ are the actual multiplication matrices, they commute because P/I is a commutative ring. Surprisingly, the converse holds, too.

Theorem 1.2.8. *(Mourrain [26])*
The set G is the $\mathcal{O}$-border basis of I if and only if the formal multiplication matrices commute, i.e. iff

$$\mathcal{A}_i \mathcal{A}_j = \mathcal{A}_j \mathcal{A}_i \qquad \text{for } 1 \le i < j \le n.$$

For a detailed proof, see [21], 6.4.30. Let us check this result in a concrete case.

Example 1.2.9. In Example 1.2.2 the multiplication matrices are given by

$$\mathcal{A}_x = \begin{pmatrix} 0 & 0.8 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } \mathcal{A}_y = \begin{pmatrix} 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.8 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

To check this, we let $t_1 = 1$, $t_2 = x$, $t_3 = y$, and $t_4 = xy$. Then we note that for instance $xt_1 = t_2$ means that the first column of $\mathcal{A}_x$ is $(0,1,0,0)$. If we compute $xt_2 = x^2$, we have to use the coefficients of the corresponding border prebasis polynomial g_1 and put $(0.8,0,0,0)$ into the second column of $\mathcal{A}_1$, etc.

1.2.C. Neighbors and their syzygies. Our next goal is to generalize the Buchberger Criterion for Gröbner bases (see [20], 2.5.3) to the border basis setting. The Buchberger criterion is based on the notion of *lifting syzygies*. Given an order ideal $\mathcal{O} = \{t_1,\dots,t_\mu\}$ and its border $\{b_1,\dots,b_\nu\}$, it is well-known that the syzygy module

$$\mathrm{Syz}_P(b_1,\dots,b_\nu) = \{(f_1,\dots,f_\nu \in P^\nu \mid f_1 b_1 + \cdots + f_\nu b_\nu = 0\}$$

is generated by the **fundamental syzygies**

$$\sigma_{ij} = (\mathrm{lcm}(b_i,b_j)/b_i)\, e_i - (\mathrm{lcm}(b_i,b_j)/b_j)\, e_j$$

with $1 \leq i < j \leq \nu$. However, this system of generators is not minimal and a much smaller subset suffices to generate the same module. The following terminology will be useful to describe such as subset.

Definition 1.2.10. Let $b_i, b_j \in \partial\mathscr{O}$ be two distinct border terms.

a) The border terms b_i and b_j are called **next-door neighbors** if $b_i = x_k b_j$ for some $k \in \{1,\ldots,n\}$.
b) The border terms b_i and b_j are called **across-the-street neighbors** if there are $k,\ell \in \{1,\ldots,n\}$ such that $x_k b_i = x_\ell b_j$.
c) The border terms b_i and b_j are called **neighbors** if they are next-door neighbors or across-the-street neighbors.
d) The graph whose vertices are the border terms and whose edges are given by the neighbor relation is called the **border web** of $\mathscr{O}$.

Example 1.2.11. For instance, in Example 1.2.2 the border is $\partial\mathscr{O} = \{b_1, b_2, b_3, b_4\}$ with $b_1 = x^2$, $b_2 = x^2y$, $b_3 = xy^2$, and $b_4 = y^2$. Here we have two next-door neighbor pairs (b_2, b_1), (b_3, b_4) and one across-the-street neighbor pair (b_2, b_3).

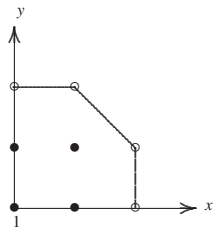


Fig. 1.5 A simple border web.

Proposition 1.2.12. *The border web is connected.*

For a proof, see [17], Prop. 19. Based on the concept of neighbors, we now restrict fundamental syzygies to neighbor pairs.

Definition 1.2.13. Let $\mathscr{O}$ be an order ideal with border $\partial\mathscr{O} = \{b_1,\ldots,b_\nu\}$.

a) For next-door neighbors b_i, b_j, i.e. for $b_i = x_k b_j$, the fundamental syzygy σ_{ij} has the form $\tau_{ij} = e_i - x_k e_j$ and is called a **next-door neighbor syzygy**.
b) For across-the-street neighbors b_i, b_j, i.e. for $x_k b_i = x_\ell b_j$, the fundamental syzygy σ_{ij} has the form $\upsilon_{ij} = x_k e_i - x_\ell e_j$ and is called an **across-the-street neighbor syzygy**.
c) The set of all **neighbor syzygies** is the set of all next-door or across-the street neighbor syzygies.

In [17], Prop. 21, the following result is shown.

Proposition 1.2.14. *The set of neighbor syzygies generates the module of* **border syzygies** $\mathrm{Syz}_P(b_1,\ldots,b_\nu)$.

Example 1.2.15. For instance, let us compute the border syzygies for the order ideal $\mathcal{O} = \{1, x, y, xy\}$. We have $\partial\mathcal{O} = \{b_1, b_2, b_3, b_4\}$ with $b_1 = x^2$, $b_2 = x^2y, b_3 = xy^2$, and $b_4 = y^2$, and the neighbor pairs (b_1, b_2), (b_2, b_3), (b_3, b_4). Therefore the border syzygy module $\mathrm{Syz}_P(b_1, b_2, b_3, b_4)$ is generated by the following three neighbor syzygies:

$$\begin{aligned} e_2 - y e_1 &= (-y, 1, 0, 0) \\ y e_2 - x e_3 &= (0, y, -x, 0) \\ e_4 - x e_3 &= (0, 0, -x, 1) \end{aligned}$$

In order to transfer the Buchberger Criterion from Gröbner to border bases, it suffices to lift neighbor syzygies.

Definition 1.2.16. Let $g_i, g_j \in G$ be two distinct border prebasis polynomials. Then the polynomial

$$S_{ij} = (\mathrm{lcm}(b_i, b_j)/b_i) \cdot g_i - (\mathrm{lcm}(b_i, b_j)/b_j) \cdot g_j$$

is called the **S-polynomial** of g_i and g_j.

Remark 1.2.17. Let $g_i, g_j \in G$.

a) If (b_i, b_j) are next-door neighbors with $b_j = x_k b_i$ then the S-polynomial S_{ij} is of the form $S_{ij} = g_j - x_k g_i$.
b) If (b_i, b_j) are across-the-street neighbors with $x_k b_i = x_\ell b_j$ then S_{ij} is of the form $S_{ij} = x_k g_i - x_\ell b_j$.

In both cases we see that the support of S_{ij} is contained in $\mathcal{O} \cup \partial\mathcal{O}$. Hence there exists constants $a_i \in K$ such that the support of

$$\mathrm{NR}_{\mathcal{O},G}(S_{ij}) = S_{ij} - \sum_{m=1}^{\mu} a_m g_m \in I$$

is contained in $\mathcal{O}$. If G is a border basis, this implies $\mathrm{NR}_{\mathcal{O},G}(S_{ij}) = 0$. We shall say that the syzygy $e_j - x_k e_i - \sum_{m=1}^{\mu} a_m e_m$ resp. $x_k\, e_i - x_\ell e_j - \sum_{m=1}^{\mu} a_m e_m$ is a **lifting** of the neighbor syzygy $e_j - x_k e_i$ resp. $x_k\, e_i - x_\ell e_j$.

Theorem 1.2.18. *(Stetter [30])*
An $\mathcal{O}$-border prebasis G is an $\mathcal{O}$-border basis if and only if the neighbor syzygies lift, i.e. if and only if we have

$$\mathrm{NR}_{\mathcal{O},G}(S_{ij}) = 0$$

for all (i, j) such that (b_i, b_j) is a pair of neighbors.

The proof of this theorem is pretty involved. Let us briefly describe the idea. The vanishing conditions for the normal remainders of the S-polynomials entail certain equalities which have to be satisfied by the coefficients c_{ij} of the border prebasis polynomials. Using a (rather nasty) case-by-case argument, one checks that these are the same equalities that one gets from the conditions that the formal multiplication matrices have to commute. A detailed version of this proof is contained in [21], Section 6.4.

Example 1.2.19. Let us look at these conditions for $\mathcal{O} = \{1, x, y, xy\}$. An $\mathcal{O}$-border prebasis $G = \{g_1, g_2, g_3, g_4\}$ is of the form

$$\begin{aligned} g_1 &= x^2 - c_{11} \cdot 1 - c_{21}\, x - c_{31}\, y - c_{41}\, xy \\ g_2 &= x^2 y - c_{12} \cdot 1 - c_{22}\, x - c_{32}\, y - c_{42}\, xy \\ g_3 &= xy^2 - c_{13} \cdot 1 - c_{23}\, x - c_{33}\, y - c_{43}\, xy \\ g_4 &= y^2 - c_{14} \cdot 1 - c_{24}\, x - c_{34}\, y - c_{44}\, xy \end{aligned}$$

The S-polynomials of its neighbor syzygies are

$$\begin{aligned} S_{21} &= g_2 - y g_1 \\ &= -c_{12} - c_{22} x + (c_{11} - c_{32}) y + (c_{21} - c_{42}) xy + c_{31} y^2 + c_{41} xy^2 \\ S_{23} &= y g_2 - x g_3 \\ &= c_{13} x - c_{22} y + (c_{33} - c_{22}) xy + c_{23} x^2 + c_{43} x^2 y - c_{42} xy^2 - c_{32} y^2 \\ S_{34} &= g_3 - x g_4 \\ &= -c_{13} + (c_{14} - c_{23}) x - c_{33} y + (c_{34} - c_{43}) xy + c_{24} x^2 + c_{44} x^2 y \end{aligned}$$

Their normal remainders with respect to G are

$$\begin{aligned} \mathrm{NR}_{\mathcal{O},G}(S_{21}) &= (-c_{12} + c_{31} c_{14} + c_{41} c_{13}) + (-c_{22} + c_{31} c_{24} + c_{41} c_{23}) x \\ &\quad + (c_{11} - c_{32} + c_{31} c_{34} + c_{41} c_{33}) y + (c_{21} - c_{42} + c_{31} c_{44} + c_{41} c_{43}) xy \\ \mathrm{NR}_{\mathcal{O},G}(S_{23}) &= (c_{11} c_{23} + c_{12} c_{43} - c_{42} c_{13} - c_{32} c_{14}) + (c_{21} c_{23} + c_{22} c_{43} \\ &\quad - c_{42} c_{23} - c_{32} c_{24} + c_{13}) x + (-c_{12} + c_{31} c_{23} + c_{32} c_{43} - c_{42} c_{33} - c_{32} c_{34}) y \\ &\quad + (c_{33} - c_{22} + c_{41} c_{23} - c_{32} c_{44}) xy \\ \mathrm{NR}_{\mathcal{O},G}(S_{34}) &= (-c_{13} + c_{11} c_{24} + c_{12} c_{44}) + (c_{14} - c_{23} + c_{21} c_{24} + c_{23} c_{44}) x \\ &\quad + (-c_{33} + c_{31} c_{24} + c_{32} c_{44}) y + (c_{34} - c_{43} + c_{41} c_{24} + c_{42} c_{44}) xy \end{aligned}$$

Here G is a border basis if and only if these 12 coefficients are zero. In Example 1.6.14 we shall examine the scheme defined by these vanishing conditions.

1.3 The Eigenvalue Method for Solving Polynomial Systems

When working toward the solution of a problem,
it always helps if you know the answer.
(Rule of Accuracy)

As said in the introduction, this paper deals mainly with the problem of reconstructing polynomial equations from data. The *opposite problem* of solving polynomial systems is also well-known since it plays a key role in many applications. Rather than trying to discuss this problem in its full generality, we will now have a look at a nice method which deserves to be more widely known in the commutative algebra community. While the so called Lex-method is amply described in the literature (see for instance [20], Section 3.7), we are going to describe an idea on how to use classical methods in linear algebra to solve polynomial systems. The pioneering work was done in [4] and [5], and a nice introduction can be found in [9].

In the following we let K be a field and $P = K[x_1, \ldots, x_n]$. Let the polynomial system be defined by $f_1, \ldots, f_s \in P$. Then we let $I = (f_1, \ldots, f_s)$ and $A = P/I$. We assume that the ideal I is zero-dimensional, so that A is a finite dimensional K-vector space.

Definition 1.3.1. Given an element f in P, we define a K-linear map $m_f : A \longrightarrow A$ by $m_f(g) = fg \mod I$ and call it the **multiplication map** defined by f. We also consider the induced K-linear map on the dual spaces $m_f^* : A^* \longrightarrow A^*$ defined by $m_f^*(\varphi) = \varphi \circ m_f$.

If we know a vector space basis of A, we can represent a multiplication map by its matrix with respect to this basis. Let us have a look at a concrete case.

Example 1.3.2. Let $P = \mathbb{R}[x]$, let $f = x^2 + 1$, and I be the principal ideal generated by $x^3 - x^2 + x - 1 = (x-1)(x^2+1)$. The residue classes of the terms in $\{1, x, x^2\}$ form a vector space basis of P/I. Using the two relations $x^3 + x \equiv x^2 + 1 \mod I$ and $x^4 + x^2 \equiv x^2 + 1 \mod I$, we see that the matrix which represents m_f with respect to this basis is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

The next theorem provides an important link between $\mathscr{Z}(I)$, the set of *zeros* of I over the algebraic closure $\overline{K}$ of K, and the eigenvalues of multiplication maps. Let $\overline{P} = \overline{K}[x_1, \ldots, x_n]$ and $\overline{A} = \overline{P}/I\overline{P}$. An element $\lambda \in \overline{K}$ is called a **$\overline{K}$-eigenvalue** of a multiplication map $m_f : A \longrightarrow A$ if it is a zero of the **characteristic polynomial** of m_f, i.e. if the $\overline{K}$-linear map $\varphi_{f,\lambda} : \overline{A} \longrightarrow \overline{A}$ defined by $\bar{g} \mapsto \bar{f}\bar{g} - \lambda\,\bar{g}$ is not invertible.

Theorem 1.3.3. *Let I be a zero-dimensional ideal in P, let $f \in P$, and let $\lambda \in \overline{K}$, Then the following conditions are equivalent.*

a) The element λ is a $\overline{K}$-eigenvalue of m_f.

b) There exists a point $p \in \mathscr{Z}(I)$ *such that* $\lambda = f(p)$.

In this equivalence, if $p \in \mathscr{Z}_K(I)$*, we have* $\lambda \in K$.

Proof. Let us first prove $a) \implies b)$. If λ does not coincide with any of the values of f at the points $p \in \mathscr{Z}(I)$, the ideal $J = I\overline{P} + (f-\lambda) \subseteq \overline{P}$ satisfies $\mathscr{Z}(J) = \emptyset$. Thus the Weak Nullstellensatz (see [20], Corollary 2.6.14) yields $1 \in J$. Therefore there exist $g \in \overline{P}$ and $h \in I\overline{P}$ such that $1 = g(f-\lambda) + h$. Consequently, we have $1 \equiv g(f-\lambda) \mod I\overline{P}$, so that $\varphi_{f,\lambda}$ is invertible with inverse $-m_{\bar{g}}$. Therefore λ is not a $\overline{K}$-eigenvalue of m_f.

Now let us prove the implication $b) \implies a)$. If λ is not a $\overline{K}$-eigenvalue of m_f then $\varphi_{f,\lambda}$ is an invertible map. In particular, it is surjective, and thus there exists $g \in \overline{P}$ such that $g(f-\lambda) \equiv 1 \mod I\overline{P}$. Clearly, this implies that there cannot exist a point $p \in \mathscr{Z}(I)$ such that $f(p) - \lambda = 0$.

The additional claim follows from $\lambda = f(p)$. □

In the setting of Example 1.3.2, the eigenvalues of m_f and the zeros of I are related as follows.

Example 1.3.4. As in Example 1.3.2, we let $I = (x^3 - x^2 + x - 1) \subseteq P = \mathbb{R}[x]$ and $f = x^2 + 1$. Since m_f is singular, the element $\lambda_1 = 0$ is an eigenvalue. And indeed, we have $\mathscr{Z}(I) = \{1, i, -i\}$ and $f(i) = f(-i) = 0$. For the other eigenvalue $\lambda_2 = 2$, we have $f(1) = 2$. Notice that here we have $\lambda_1 \in \mathbb{R}$, but the corresponding zeros of I are not real numbers.

The above theorem can be used in several ways to compute the solutions of a system of polynomial equations. One method is based on the following observation.

Corollary 1.3.5. *Let* $i \in \{1, \ldots, n\}$. *The* i^{th} *coordinates of the points of* $\mathscr{Z}(I)$ *are the* $\overline{K}$*-eigenvalues of the multiplication map* m_{x_i}.

Proof. This follows immediately from the theorem, since $x_i(p)$ is exactly the i^{th} coordinate of a point $p \in \overline{K}^n$. □

Hence we can determine $\mathscr{Z}(I)$ in the following way. Fix a tuple of polynomials $E = (t_1, \ldots, t_\mu)$ whose residue classes form a K-basis of A. For $f \in P$, we let $fE = (ft_1, \ldots, ft_\mu)$ and describe the multiplication map $m_f : A \longrightarrow A$ by the matrix M_{fE}^E whose the j^{th} column $(a_{1j}, \ldots, a_{\mu j})^{\text{tr}}$ is given by

$$f t_j \equiv a_{1j} t_1 + \cdots + a_{\mu j} t_\mu \mod I$$

A more compact way of expressing this fact is the formula

$$fE \equiv E \cdot M_{fE}^E \mod I \qquad (*)$$

For the tuple E, we usually choose an order ideal of terms (see Definition 1.2.1). In particular, we shall assume that we have $t_1 = 1$.

If the ideal I contains a linear polynomial, we can reduce the problem of computing $\mathscr{Z}(I)$ to a problem for an ideal in a polynomial ring having fewer indeterminates. Thus we shall now assume that I contains no linear polynomial. Consequently, we suppose that the indeterminates are in E, specifically that we have $t_2 = x_1, \ldots, t_{n+1} = x_n$.

One method of finding $\mathscr{Z}(I)$ is to compute the $\overline{K}$-eigenvalues $\lambda_{i1}, \ldots, \lambda_{i\mu}$ of $M^E_{x_iE}$ for $i = 1, \ldots, n$ and then to check for all points $(\lambda_{1j_1}, \ldots, \lambda_{nj_n})$ such that $j_1, \ldots, j_n \in \{1, \ldots, \mu\}$ whether they are zeros of I. Clearly, this approach has several disadvantages:

1. Usually, the $\overline{K}$-eigenvalues of the multiplication matrices $M^E_{x_iE}$ can only be determined approximatively.
2. The set of candidate points is a grid which is typically much larger than the set $\mathscr{Z}(I)$.

A better approach uses the next theorem. For a K-linear map $\varphi : A \longrightarrow A$, we let $\bar{\varphi} = \varphi \otimes_K \overline{K} : \overline{A} \longrightarrow \overline{A}$. Given a $\overline{K}$-eigenvalue $\lambda \in \overline{K}$ of φ, the $\overline{K}$-vector space $\ker(\bar{\varphi} - \lambda \, \mathrm{id}_{\overline{A}})$ is called the corresponding **$\overline{K}$-eigenspace** and its non-zero vectors are called the corresponding **$\overline{K}$-eigenvectors**. For the matrices representing φ, we use a similar terminology.

Theorem 1.3.6. *In the above setting, let $f \in P$, let $p \in \mathscr{Z}(I)$, and let $E = (t_1, \ldots, t_\mu)$ be a tuple of polynomials whose residue classes form a K-basis of A. Then the vector $E(p)^{\mathrm{tr}} = (t_1(p), \ldots, t_\mu(p))^{\mathrm{tr}}$ is a $\overline{K}$-eigenvector of $(M^E_{fE})^{\mathrm{tr}}$ corresponding to the $\overline{K}$-eigenvalue $f(p)$.*

Proof. When we evaluate both sides of the above formula $(*)$ at p, we get the equality $f(p)E(p) = E(p)M^E_{fE}$. Transposing both sides yields

$$f(p)(E(p))^{\mathrm{tr}} = (M^E_{fE})^{\mathrm{tr}} E(p)^{\mathrm{tr}}$$

and this is precisely the claim. □

Note that the matrix $(M^E_{fE})^{\mathrm{tr}}$ represents the linear map m^*_f (see Definition 1.3.1). To make good use of this theorem, we need the following notion.

Definition 1.3.7. A matrix $M \in \mathrm{Mat}_\mu(K)$ is called **$\overline{K}$-non-derogatory** if it has the property that all its $\overline{K}$-eigenspaces are 1-dimensional.

It is a well-known result in Linear Algebra that this condition is equivalent to requiring that the Jordan canonical form of M over $\overline{K}$ has one Jordan block per eigenvalue, or to the condition that the minimal polynomial and the characteristic polynomial of M agree. Using the preceding theorem and a non-derogatory multiplication matrix, we can solve a zero-dimensional polynomial system as follows.

Corollary 1.3.8. *Let $E = (t_1, \ldots, t_\mu)$ be a tuple of polynomials whose residue classes form a K-basis of A, let $1, x_1, \ldots, x_n$ be the first $n+1$-entries of E, and let $f \in P$ be such that the matrix $(M^E_{fE})^{\mathrm{tr}}$ is $\overline{K}$-non-derogatory. Let $V_1, \ldots, V_r$ be*

the $\overline{K}$-eigenspaces of this matrix. For $j = 1,\dots,r$, choose a basis vector v_j of V_j of the form $v_j = (1, a_{2j}, \dots, a_{\mu j})$ with $a_{ij} \in \overline{K}$. Then $\mathscr{Z}(I)$ consists of the points $p_j = (a_{2j}, \dots, a_{n+1\,j})$ such that $j \in \{1,\dots,r\}$.

Proof. Let $j \in \{1,\dots,r\}$. By Theorem 1.3.6, the vector $E(p_j) = (t_1(p_j),\dots,t_\mu(p_j))$ is a $\overline{K}$-eigenvector of $(M_{fE}^E)^{\mathrm{tr}}$ corresponding to the $\overline{K}$-eigenvalue $f(p_j)$. Hence it is a non-zero vector in V_j. Since V_j is 1-dimensional and $t_1(p_j) = 1$ equals the first component of v_j, we have the equality $E(p_j) = v_j$. Now the observation that $E(p_j) = (1, x_1(p_j), \dots, x_n(p_j), \dots)$ finishes the proof. □

The technique given in this corollary addresses the second problem stated above: no exponentially large set of candidate points has to be examined. However, we note that the first problem still persists. For instance, if $K = \mathbb{Q}$, instead of the $\overline{\mathbb{Q}}$-eigenvalues of $(M_{fE}^E)^{\mathrm{tr}}$ we can usually only compute approximate eigenvalues. Hence the corresponding $\overline{\mathbb{Q}}$-eigenspaces are not computable as true kernels of linear maps. However, in the next section we will introduce *approximate kernels* of linear maps which take care of this task.

Let us end this section with an example which illustrates the methods described above.

Example 1.3.9. Let I be the ideal in $P = \mathbb{R}[x,y]$ generated by the set of polynomials $\{x^2 + 4/3xy + 1/3y^2 - 7/3x - 5/3y + 4/3,\ y^3 + 10/3xy + 7/3y^2 - 4/3x - 20/3y + 4/3,\ xy^2 - 7/3xy - 7/3y^2 - 2/3x + 11/3y + 2/3\}$. It is easy to check that this set is a Gröbner basis of I with respect to $\sigma = \texttt{DegRevLex}$. Hence $E = \{1, x, y, xy, y^2\}$ is an order ideal of terms whose residue classes form a K-basis of P/I. By computing the normal forms $\mathrm{NF}_{\sigma,I}(x^2)$, $\mathrm{NF}_{\sigma,I}(x^2y)$, $\mathrm{NF}_{\sigma,I}(xy^2, I)$ and $\mathrm{NF}_{\sigma,I}(y^3)$, we get the multiplication matrices

$$M_{xE}^E = \begin{pmatrix} 0 & -4/3 & 0 & 4/3 & -2/3 \\ 1 & 7/3 & 0 & -4/3 & 2/3 \\ 0 & 5/3 & 0 & 4/3 & -11/3 \\ 0 & -4/3 & 1 & 1/3 & 7/3 \\ 0 & -1/3 & 0 & -2/3 & 7/3 \end{pmatrix} \text{ and } M_{yE}^E = \begin{pmatrix} 0 & 0 & 0 & -2/3 & -4/3 \\ 0 & 0 & 0 & 2/3 & 4/3 \\ 1 & 0 & 0 & -11/3 & 20/3 \\ 0 & 1 & 0 & 7/3 & -10/3 \\ 0 & 0 & 1 & 7/3 & -7/3 \end{pmatrix}$$

First, let us follow the method of Corollary 1.3.5. The characteristic polynomial of M_{xE}^E is $(x+1)(x-1)^2(x-2)^2$ and the characteristic polynomial of M_{yE}^E is given by $x(x-1)(x+1)(x-2)(x+2)$. If we check the 15 candidate points, we find that five of them, namely $(1,0)$, $(1,1)$, $(2,-1)$, $(-1,2)$, and $(2,-2)$ form the set of zeros of I.

Now we apply the method of Corollary 1.3.8. The characteristic polynomial of $(M_{xE}^E)^{\mathrm{tr}}$ is the same as that of M_{xE}^E. It is easy to check (for instance, using CoCoA) that the dimension of the eigenspace corresponding to the eigenvalue 1 is 2. Therefore the matrix $(M_{xE}^E)^{\mathrm{tr}}$ is derogatory and cannot be used for the proposed method.

On the other hand, the characteristic polynomial of the matrix $(M_{yE}^E)^{\mathrm{tr}}$ is given by $x(x-1)(x+1)(x-2)(x+2)$. Consequently, this matrix is non-derogatory. We compute basis vectors for its eigenspaces and norm them to have first component 1.

The result is $v_1 = (1,1,0,0,0)$, $v_2 = (1,1,1,1,1)$, $v_3 = (1,2,-1,-2,1)$, $v_4 = (1,-1,2,-2,4)$, and $v_5 = (1,2,-2,-4,4)$. We get $\mathscr{L}(I) = \{(1,0), (1,1), (2,-1), (-1,2), (2,-2)\}$, as before.

1.4 Approximate Vanishing Ideals

Two is not equal to three;
not even for large values of two.
(Grabel's Law)

It is time to enter the *real world*. When dealing with industrial applications, we do not always have exact data available. Thus our computations have to be based on measured values with intrinsic errors. How can we perform symbolic computation in this world? Let us start to discuss this question in a first relevant case. Then, based on our answer, we shall present an actual industrial example. We want to deal with the following situation. Let $\mathbb{X} = \{p_1, \ldots, p_s\}$ be a set of s points in $\mathbb{R}^n$. These points are meant to represent measured values. In the computer, they will be stored as tuples of floating point numbers.

If $\mathbb{X}$ was an exact set of points, we could compute its **vanishing ideal**

$$I(\mathbb{X}) = \{f \in \mathbb{R}[x_1, \ldots, x_n] \mid f(p_1) = \cdots = f(p_s) = 0\}$$

However, in the presented setting, it is well-known that this leads to a numerically unstable and virtually meaningless result. Instead, we are looking for a reasonable definition of an **approximate vanishing ideal** of $\mathbb{X}$. To this end, we have to overcome a number of impediments. First of all, we need a **threshold number** $\varepsilon \in \mathbb{R}_+$. We say that a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ **vanishes ε-approximately** at $\mathbb{X}$ if $|f(p_i)| < \varepsilon$ for $i = 1, \ldots, s$. This definition entails several problems.

1. The polynomials which vanish ε-approximately at $\mathbb{X}$ do not form an ideal!
2. All polynomials with very small coefficients vanish ε-approximately at $\mathbb{X}$!

To address the second problem, we introduce a topology on the polynomial ring $P = \mathbb{R}[x_1, \ldots, x_n]$.

Definition 1.4.1. Let $f = a_1 t_1 + \cdots + a_s t_s \in P$, where $a_1, \ldots, a_s \in \mathbb{R} \setminus \{0\}$ and $t_1, \ldots, t_s \in \mathbb{T}^n$. Then the number $\|f\| = \|(a_1, \ldots, a_s)\| = \sqrt{a_1^2 + \cdots + a_s^2}$ is called the **(Euclidean) norm** of f.

Clearly, this definition turns P into a normed vector space. A polynomial $f \in P$ with $\|f\| = 1$ will be called **unitary**. Now it is reasonable to consider the condition that polynomials $f \in P$ with $\|f\| = 1$ vanish ε-approximately at $\mathbb{X}$, and we can try the following definition.

Definition 1.4.2. An ideal $I \subseteq P$ is called an **ε-approximate vanishing ideal** of $\mathbb{X}$ if there exists a system of generators $\{f_1, \ldots, f_r\}$ of I such that $\|f_i\| = 1$ and f_i vanishes ε-approximately at $\mathbb{X}$ for $i = 1, \ldots, r$.

In itself, this definition is certainly still too loose. For instance, it is clear that the unit ideal is always an ε-approximate vanishing ideal. Nevertheless, we shall see below that we arrive at a very usable definition if we impose additional structure on the generators. Before we move to this topic, though, we need two additional ingredients.

1.4.A. The Singular Value Decomposition (SVD). In approximate computation, we frequently have to decide whether something is zero or not. The following theorem and its corollary can be used to determine the vectors which are approximately in the kernel of a linear map of $\mathbb{R}$-vector spaces.

Theorem 1.4.3 (The Singular Value Decomposition).
Let $\mathscr{A} \in \mathrm{Mat}_{m,n}(\mathbb{R})$.

1. *There are orthogonal matrices $\mathscr{U} \in \mathrm{Mat}_{m,m}(\mathbb{R})$ and $\mathscr{V} \in \mathrm{Mat}_{n,n}(\mathbb{R})$ and a matrix $\mathscr{S} \in \mathrm{Mat}_{m,n}(\mathbb{R})$ of the form $\mathscr{S} = \begin{pmatrix} \mathscr{D} & 0 \\ 0 & 0 \end{pmatrix}$ such that*

$$\mathscr{A} = \mathscr{U} \cdot \mathscr{S} \cdot \mathscr{V}^{\mathrm{tr}} = \mathscr{U} \cdot \begin{pmatrix} \mathscr{D} & 0 \\ 0 & 0 \end{pmatrix} \cdot \mathscr{V}^{\mathrm{tr}}$$

   where $\mathscr{D} = \mathrm{diag}(s_1, \ldots, s_r)$ is a diagonal matrix.
2. *In this decomposition, it is possible to achieve $s_1 \geq s_2 \geq \cdots \geq s_r > 0$. The numbers $s_1, \ldots, s_r$ depend only on $\mathscr{A}$ and are called the* **singular values** *of $\mathscr{A}$.*
3. *The number r is the rank of $\mathscr{A}$.*
4. *The matrices $\mathscr{U}$ and $\mathscr{V}$ have the following interpretation:*

$$\begin{aligned} \textit{first } r \textit{ columns of } \mathscr{U} &\equiv \textit{ONB of the column space of } \mathscr{A} \\ \textit{last } m-r \textit{ columns of } \mathscr{U} &\equiv \textit{ONB of the kernel of } \mathscr{A}^{\mathrm{tr}} \\ \textit{first } r \textit{ columns of } \mathscr{V} &\equiv \textit{ONB of the row space of } \mathscr{A} \\ &\equiv \textit{ONB of the column space of } \mathscr{A}^{\mathrm{tr}} \\ \textit{last } n-r \textit{ columns of } \mathscr{V} &\equiv \textit{ONB of the kernel of } \mathscr{A} \end{aligned}$$

   Here ONB is an abbreviation for "orthonormal basis".

For a proof, see for instance [12], Sections 2.5.3 and 2.6.1. The SVD of a real matrix allows us to define and compute its **approximate kernel**.

Corollary 1.4.4. *Let $\mathscr{A} \in \mathrm{Mat}_{m,n}(\mathbb{R})$, and let $\varepsilon > 0$ be given. Choose $k \in \{1, \ldots, r\}$ such that $s_k > \varepsilon \geq s_{k+1}$, form the matrix $\widetilde{\mathscr{S}}$ by setting $s_{k+1} = \cdots = s_r = 0$ in $\mathscr{S}$, and let $\widetilde{\mathscr{A}} = \mathscr{U}\,\widetilde{\mathscr{S}}\,\mathscr{V}^{\mathrm{tr}}$.*

1. *We have $\min\{\|\mathscr{A} - \mathscr{B}\| : \mathrm{rank}(\mathscr{B}) \leq k\} = \|\mathscr{A} - \widetilde{\mathscr{A}}\| = s_{k+1}$. (Here $\|\cdots\|$ denotes the 2-operator norm of a matrix.)*

2. *The vector subspace* $\operatorname{apker}(\mathscr{A},\varepsilon) = \ker(\widetilde{\mathscr{A}})$ *is the largest dimensional kernel of a matrix whose Euclidean distance from* $\mathscr{A}$ *is at most* ε. *It will be called the* ε**-approximate kernel** *of* $\mathscr{A}$.
3. *The last* $n-k$ *columns* $v_{k+1},\ldots,v_n$ *of* $\mathscr{V}$ *are an ONB of* $\operatorname{apker}(\mathscr{A},\varepsilon)$. *They satisfy* $\|\mathscr{A}v_i\| < \varepsilon$.

Proof. See [12], Section 2.5.4 and the theorem. To prove the third claim, observe that $\|\mathscr{A}v_i\| = \|(\mathscr{A}-\widetilde{\mathscr{A}})v_i\| \leq \|\mathscr{A}-\widetilde{\mathscr{A}}\| < \varepsilon$. □

1.4.B. The Stable Reduced Row Echelon Form. Our next task is to find the leading terms contained in a vector space of polynomials. Again we are of course interested in leading terms of unitary polynomials for which the leading coefficient is not smaller than a given threshold number.

Let $V \subset P$ be a finite dimensional vector space of polynomials. Given a term ordering σ and a basis $B = \{f_1,\ldots,f_r\}$ of V, We can identify V with a real matrix as follows.

Definition 1.4.5. Let $S = \operatorname{Supp}(f_1) \cup \cdots \cup \operatorname{Supp}(f_r)$, and write $S = \{t_1,\ldots,t_s\}$ where the terms $t_i \in \mathbb{T}^n$ are ordered such that $t_1 \geq_\sigma t_2 \geq_\sigma \cdots \geq_\sigma t_s$. Clearly, the support of every polynomial of V is contained in S. For $i = 1,\ldots,r$, we write $f_i = c_{i1}t_1 + \cdots + c_{is}t_s$ with $c_{ij} \in \mathbb{R}$. Then the matrix $M_{\sigma,B} = (c_{ij}) \in \operatorname{Mat}_{r,s}(\mathbb{R})$ is called the **Macaulay matrix** of V with respect to σ and B.

In other words, the columns of $M_{\sigma,B}$ are indexed by the terms in S and the rows correspond to the coefficients of the basis polynomials f_i. If we use Gaußian elimination to bring $M_{\sigma,B}$ into row echelon form, the first non-zero entries of each row will indicate the leading term of the corresponding polynomial. Hence the pivot columns will correspond precisely to the set $\operatorname{LT}_\sigma(V)$ of all leading terms of polynomials in V.

To imitate this in the approximate world, we should perform the Gaußian elimination in a numerically stable way. However, we cannot use complete pivoting, since the order of the rows is fixed by the term ordering. The following adaptation of the QR-decomposition uses partial pivoting and provides the "best" leading terms available under the given circumstances.

Proposition 1.4.6 (Stabilized Reduced Row Echelon Form).
Let $A \in \operatorname{Mat}_{m,n}(\mathbb{R})$ *and* $\tau > 0$ *be given. Let* $a_1,\ldots,a_n$ *be the columns of* A. *Consider the following instructions.*

(1) Let $\lambda_1 = \|a_1\|$. *If* $\lambda_1 < \tau$, *we let* $R = (0,\ldots,0) \in \operatorname{Mat}_{m,1}(\mathbb{R})$. *Otherwise, we let* $Q = ((1/\lambda_1)\,a_1) \in \operatorname{Mat}_{m,1}(\mathbb{R})$ *and* $R = (\lambda_1,0,\ldots,0) \in \operatorname{Mat}_{m,1}(\mathbb{R})$.
(2) For $i = 2,\ldots,n$, *compute* $q_i = a_i - \sum_{j=1}^{i-1}\langle a_i,q_j\rangle\, q_j$ *and* $\lambda_i = \|q_i\|$. *If* $\lambda_i < \tau$, *append a zero column to* R. *Otherwise, append the column* $(1/\lambda_i)\,q_i$ *to* Q *and the column* $(\lambda_i\langle a_1,q_1\rangle,\ldots,\lambda_i\langle a_{i-1},q_{i-1}\rangle,\lambda_i,0,\ldots,0)$ *to* R.
(3) Starting with the last row and working upwards, use the first non-zero entry of each row of R *to clean out the non-zero entries above it.*

(4) For $i = 1,\dots,m$, compute the norm ρ_i of the i-th row of R. If $\rho_i < \tau$, set this row to zero. Otherwise, divide this row by ρ_i. Then return the matrix R.

This is an algorithm which computes a matrix R in reduced row echelon form. The row space of R is contained in the row space of the matrix $\overline{A}$ which is obtained from A by setting the columns whose norm is less than τ to zero. Here the pivot elements of R are not 1, but its rows are unitary vectors.

Furthermore, if the rows of A are unitary and mutually orthogonal, the row vectors of R differ by less than $\tau m\sqrt{n}$ from unitary vectors in the row space of A.

The proof of this proposition in contained in [14], Section 3.

1.4.C. The AVI-Algorithm. Finally we are ready to combine all ingredients and produce an algorithm which computes a "good" system of generators of an approximate vanishing ideal of $\mathbb{X}$. By "good" we mean the following.

Definition 1.4.7. Let $\mathscr{O} = \{t_1,\dots,t_\mu\}$ be an order ideal of terms in $\mathbb{T}^n$, denote its border by $\partial\mathscr{O} = \{b_1,\dots,b_\nu\}$, and let $G = \{g_1,\dots,g_\nu\}$ be an $\mathscr{O}$-border prebasis of the ideal $I = (g_1,\dots,g_\nu)$ in P. Recall that this means that g_j is of the form $g_j = b_j - \sum_{i=1}^{\mu} c_{ij}t_i$ with $c_{ij} \in \mathbb{R}$.

For every pair (i,j) such that b_i, b_j are neighbors in $\partial\mathscr{O}$, we compute the normal remainder $S'_{ij} = \mathrm{NR}_{\mathscr{O},G}(S_{ij})$ of the S-polynomial of g_i and g_j with respect to G. We say that G is an **ε-approximate border basis** of the ideal $I = (G)$ if we have $\|S'_{ij}\| < \varepsilon$ for all such pairs (i,j).

Given a finite set of points $\mathbb{X} = \{p_1,\dots,p_s\}$ in $\mathbb{R}^n$, the first thing one should do in every approximate computation is to normalize the data, i.e. to transform $\mathbb{X}$ such that $\mathbb{X} \subset [-1,1]^n$. Then the following algorithm computes an approximate border basis of an approximate vanishing ideal of $\mathbb{X}$.

Theorem 1.4.8 (The Approximate Vanishing Ideal Algorithm (AVI-Algorithm)).

Let $\mathbb{X} = \{p_1,\dots,p_s\} \subset [-1,1]^n \subset \mathbb{R}^n$, let $P = \mathbb{R}[x_1,\dots,x_n]$, let $\mathrm{eval}_\mathbb{X} : P \longrightarrow \mathbb{R}^s$ be the associated evaluation map $\mathrm{eval}_\mathbb{X}(f) = (f(p_1),\dots,f(p_s))$, and let $\varepsilon > \tau > 0$ be small positive numbers. Moreover, let σ be a degree compatible term ordering. Consider the following sequence of instructions.

A1 *Start with lists $G = \emptyset$, $\mathscr{O} = [1]$, a matrix $\mathscr{M} = (1,\dots,1)^{\mathrm{tr}} \in \mathrm{Mat}_{s,1}(\mathbb{R})$, and $d = 0$.*

A2 *Increase d by one and let L be the list of all terms of degree d in $\partial\mathscr{O}$, ordered decreasingly w.r.t. σ. If $L = \emptyset$, return the pair $(G,\mathscr{O})$ and stop. Otherwise, let $L = (t_1,\dots,t_\ell)$.*

A3 *Let m be the number of columns of $\mathscr{M}$. Form the matrix*

$$\mathscr{A} = (\mathrm{eval}_\mathbb{X}(t_1),\dots,\mathrm{eval}_\mathbb{X}(t_\ell),\mathscr{M}) \in \mathrm{Mat}_{s,\ell+m}(\mathbb{R}).$$

Using its SVD, calculate a matrix $\mathscr{B}$ whose column vectors are an ONB of the approximate kernel $\mathrm{apker}(\mathscr{A},\varepsilon)$.

A4 *Using Proposition 1.4.6, compute the stabilized reduced row echelon form of* $\mathscr{B}^{\mathrm{tr}}$ *with respect to the given* τ*. The result is a matrix* $\mathscr{C} = (c_{ij}) \in \mathrm{Mat}_{k,\ell+m}(\mathbb{R})$ *such that* $c_{ij} = 0$ *for* $j < \nu(i)$*. Here* $\nu(i)$ *denotes the column index of the pivot element in the* i^{th} *row of* $\mathscr{C}$*.*

A5 *For all* $j \in \{1,\dots,\ell\}$ *such that there exists a* $i \in \{1,\dots,k\}$ *with* $\nu(i) = j$ *(i.e. for the column indices of the pivot elements), append the polynomial*

$$c_{ij}t_j + \sum_{j'=j+1}^{\ell} c_{ij'}t_{j'} + \sum_{j'=\ell+1}^{\ell+m} c_{ij'}u_{j'}$$

to the list G*, where* $u_{j'}$ *is the* $(j'-\ell)^{\mathrm{th}}$ *element of* $\mathscr{O}$*.*

A6 *For all* $j = \ell, \ell-1, \dots, 1$ *such that the* j^{th} *column of* $\mathscr{C}$ *contains no pivot element, append the term* t_j *as a new first element to* $\mathscr{O}$ *and append the column* $\mathrm{eval}_{\mathbb{X}}(t_j)$ *as a new first column to* $\mathscr{M}$*.*

A7 *Using the SVD of* $\mathscr{M}$*, calculate a matrix* $\mathscr{B}$ *whose column vectors are an ONB of* $\mathrm{apker}(\mathscr{M}, \varepsilon)$*.*

A8 *Repeat steps* **A4** – **A7** *until* $\mathscr{B}$ *is empty. Then continue with step* **A2***.*

This is an algorithm which computes a pair $(G, \mathscr{O})$ *of sets* $G = \{g_1,\dots,g_\nu\}$ *and* $\mathscr{O} = \{t_1,\dots,t_\mu\}$ *with the following properties:*

a) The set G *consists of unitary polynomials which generate a* δ*-approximate vanishing ideal of* $\mathbb{X}$*, where* $\delta = \varepsilon\sqrt{\nu} + \tau\nu(\mu+\nu)$*.*

b) The set $\mathscr{O} = \{t_1,\dots,t_\mu\}$ *contains an order ideal of terms such that there is no unitary polynomial in* $\langle \mathscr{O} \rangle_K$ *which vanishes* ε*-approximately on* $\mathbb{X}$*.*

c) The set $\widetilde{G} = \{(1/\mathrm{LC}_\sigma(g))\, g \mid g \in G\}$ *is an* $\mathscr{O}$*-border prebasis.*

d) Let γ *denote the smallest absolute value of the border term coefficient of one of the polynomials* g_i*. Then the set* $\widetilde{G}$ *is an* η*-approximate border basis for* $\eta = 2\delta + 2\nu\delta^2/\gamma\varepsilon + 2\nu\delta\sqrt{s}/\varepsilon$*.*

For a proof, see [14], Section 3. Let us add some remarks on the performance of this algorithm.

1. The AVI-Algorithm follows in principle the method of the Buchberger-Möller Algorithm for computing the exact vanishing ideal of $\mathbb{X}$. However, we are not processing one term at a time, but all terms of a given degree simultaneously, in order to filter out "almost relations" among the evaluation vectors using the SVD. Of course, if these sets of terms are too large, we can partition them into smaller chunks to speed up the SVD calculation.
2. The stated bounds for δ and η are rather crude. Using a more refined analysis, they could be improved significantly. In practical examples, the behavior of the computed approximate border bases is much better than predicted by these bounds.
3. By changing the construction of the list L in step **A2** appropriately, the AVI-Algorithm can be used to compute an "approximate Gröbner basis" of an approximate vanishing ideal of $\mathbb{X}$. More precisely, the list L should be defined as all terms in $\mathbb{T}^n_d$ which are not contained in $\langle \mathrm{LT}_\sigma(G) \rangle$. Unfortunately, there is

no guarantee that the computed polynomials are close to an actual Gröbner basis. The computation of the normal remainders of the S-polynomials requires a number of reductions steps which can be very large. Therefore no bound for the size of the evaluation vectors of these normal remainders can be given. In many practical examples, however, the Gröbner basis version works fine.

4. The AVI-Algorithm can also be combined with a threshold control in order to obtain a smoother evaluation behaviour of the computed border prebasis. Details can be found in [14], Section 3.

What is an approximate $\mathcal{O}$-border basis good for? In the next subsection we shall see an actual industrial application. Moreover, given a further order ideal $\mathcal{O}'$ of the same cardinality, we can compute an approximate $\mathcal{O}'$-border basis using the technique of [18], Prop. 5. (In general, this will come at the expense of a partial loss of the quality of approximation.) Finally, we can compute a "close-by" exact $\mathcal{O}$-border basis having coefficients in $\mathbb{Q}$ via the *rational recovery* technique in [19], and this exact border basis can be used as input for "higher" algebraic operations such as the computation of syzygy modules.

1.4.D. An Application of the AVI Algorithm. Let us now return to Problems 1 and 2 discussed in Section 1. Our idea is to construct the desired polynomial $f \in P = \mathbb{R}[x_1,\ldots,x_5]$ using the AVI algorithm 1.4.8. Finding f means *explaining* the production as a linear combination of linearly independent "data series" which, in turn, depend on the evaluations of the indeterminates x_i. This implies that linear dependencies among those input combinations have to be removed first, i.e. that we have to pass to a suitable quotient modulo certain relations among the indeterminates. In the context of the AVI algorithm, where we are dealing with large uncertainties in the set of points $\mathbb{X}$, we need to consider *approximate* relations among the variables.

In summary, we are specifically looking for a polynomial $f \in P = \mathbb{R}[x_1,\ldots,x_5]$ of the form

$$f \;=\; \sum_{i=1}^{\mu} c_i t_i \;+\; g$$

where $g \in P$ is contained in an ε-approximate vanishing ideal of $\mathbb{X}$, where we have $c_i \in \mathbb{R}$, and where $\mathcal{O} = \{t_1,\ldots,t_\mu\}$ is an order ideal of monomials whose evaluation vectors at the points of $\mathbb{X}$ are almost linearly independent. The evaluation vector of f should represent the production data used for the modeling experiment.

Why do we expect that such a representation exists? Observe that the order ideal $\mathcal{O} = \{t_1,\ldots,t_\mu\}$ is the one calculated by the AVI algorithm. Its evaluation vectors $\{\mathrm{eval}_{\mathbb{X}}(t_1),\ldots,\mathrm{eval}_{\mathbb{X}}(t_\mu)\}$ span approximately the vector space of all possible evaluation vectors of terms at $\mathbb{X}$. Moreover, this agrees with the assumption that we tried to motivate in Section 1. Our method to compute f is to take its evaluation vector $\mathrm{eval}_{\mathbb{X}}(f)$, the measured production, and to project it to the linear span of the evaluation vectors $\mathrm{eval}_{\mathbb{X}}(t_i)$.

The results of trying this method using actual industrial data are shown in the following figure. The values of the physical quantities associated to $x_1,\ldots,x_5$ were available at 7400 time stamps. The first 6000 data points were used for the modeling

experiment, and the computed polynomial f was evaluated at the remaining 1400 data points in the validation phase. The physical interpretation of the indeterminates is according to Table 1.1.

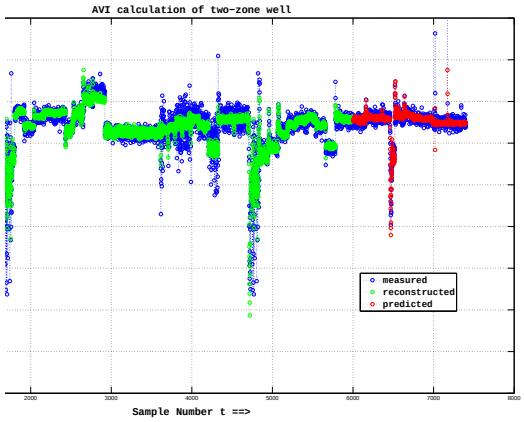


Fig. 1.6 Result of an AVI application.

Figure 1.6 shows that our model f does an excellent job: the comparison of the predicted values for the production with the measured values shows that the model passes the validation test unambiguously. The spikes shown in the figure result from instrumentation sensor failures.

Which choices and *a priori* information went into this computation? The term ordering we used is `DegRevLex`. The significance of this choice will be discussed below. For the threshold value ε, we used $\varepsilon = 0.1$. A suitable value for ε cannot be inferred from inspecting the measured data. As a rule-of-thumb, we choose it according to the size of the relative measurement errors, but we do not know a mathematical argument to determine a judicious choice of this parameter. In more intuitive terms, the value for ε is related to the level of detail we are looking at the physical problem we are investigating. In loose terms, when choosing relatively large values for ε, we are only following the large trends in the data, whereas when choosing smaller values for ε, we are zooming in on the "local" variations in the data.

Now we address one of the most remarkable features of the AVI algorithm, namely that it extracts structural information from numerical, measured data. This is unlike virtually any other method which is used in applications where a model structure in whatever form has to be provided as *input* for the algorithm which is used. Using the AVI algorithm, the model structure is *output*. Specifying a model structure up front means that a prescription is imposed on the data how the physical system under investigation works. But specifically in the case of an oil reservoir one

cannot know how it works. To emphasize this crucial point we have summarized this unique feature of the AVI algorithm in Figure 1.7.

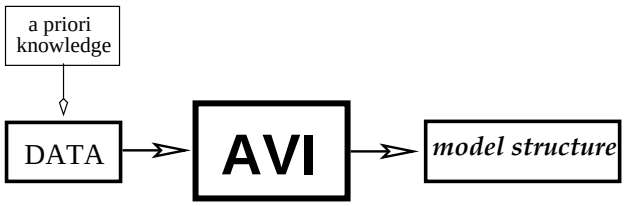


Fig. 1.7 Model construction using the AVI algorithm.

The motivation we have given in Section 1 for our problem statements indicates that a good numerical approximation and prediction of production values is not enough to deal completely with the production problem. In itself, the computed model does not give information about the all-determining interactions occurring in a production unit. For that we need to inspect the structure of the model for the production in terms of the driving inputs. In other words, we have to study the structure of the polynomial f. A first representation is

$$\begin{aligned} f = & -1.97x_3^2 - 0.18x_1x_4 - 0.30x_2x_4 + 2.37x_3x_4 - 0.16x_4^2 - 0.36x_1x_5 \\ & +0.40x_2x_5 - 3.03x_3x_5 - 1.19x_4x_5 + 0.32x_5^2 + 0.34x_1 - 0.09x_2 \\ & +4.03x_3 + 0.94x_4 + 0.68x_5 - 0.36 \end{aligned}$$

Already at first glance we notice the dominant presence of x_5. As given in Table 1.1, this indeterminate is related to the transport of the fluids through the tubing at the surface. This comes rather unexpected, indeed almost as an unpleasant surprise. For we have stated repeatedly the importance of the sub-surface for this production problem, and also that the notorious interactions are taking place in the neighborhood of the inflow from the reservoir into the production tubing of the well. But now it seems an indeterminate related to the surface, far away from the reservoir, is a key element in all this? Well, this is the answer we instructed the algorithm to find! Recall that the chosen term ordering is `DegRevLex`. Hence x_5 is the indeterminate which is most unlikely to be a leading term of one of the polynomials in the approximate border basis. In other words, it is the indeterminate which is most likely to occur in many of the terms of $\mathscr{O}$, and our method amounts to the attempt to explain the data predominantly in terms of x_5.

Rather than continuing the attempt to reveal the significance of the above structure of f, we should therefore hasten to repair our "physical inconsistency". To do this, we have two options: we can either switch to a different term ordering or we can change the physical interpretation of the indeterminates. To ease the comparison of the two models we get, we opt for the second method. The following physical interpretation of the indeterminates acknowledges the "physical hierarchy" of the system. We consider the polynomial ring $\mathbb{R}[y_1,\ldots,y_5]$ and let

$$
\begin{aligned}
y_1 &: \Delta P_{transport} \\
y_2 &: \Delta P_{tub} \\
y_3 &: \textit{Gas production} \\
y_4 &: \Delta P_{inflow_2} \\
y_5 &: \Delta P_{inflow_1}
\end{aligned}
$$

Table 1.2 New physical interpretation of the indeterminates.

For the role played by these indeterminates in the two-zone well we refer to Figure 1.1. We repeat the calculation using the AVI algorithm with $\varepsilon = 0.1$ and term ordering `DegRevLex`. The result is a polynomial g of the form

$$
\begin{aligned}
g = &-5.35y_3y_5^2 - 0.73y_4y_5^2 - 0.21y_5^3 + 2.37y_2y_3 - 7.32y_3^2 - 0.88y_1y_4 - 0.15y_2y_4 \\
&+0.34y_3y_4 - 0.55y_4^2 - 2.20y_1y_5 - 0.35y_2y_5 + 3.85y_3y_5 + 0.67y_4y_5 + 0.61y_5^2 \\
&+0.62y_1 - 0.26y_2 + 2.69y_3 + 0.98y_4 + 1.63y_5 - 0.12
\end{aligned}
$$

To judge the quality of this new model, we consider the differences of the evaluations of f and g at the points of $\mathbb{X}$. We obtain the following Figure 1.8 which shows that $f - g$ vanishes approximately at $\mathbb{X}$, apart from some spikes caused by faults in the data due to instrumentation failures. Thus, from the numerical point of view, the polynomial g is as good a model of the system as f.

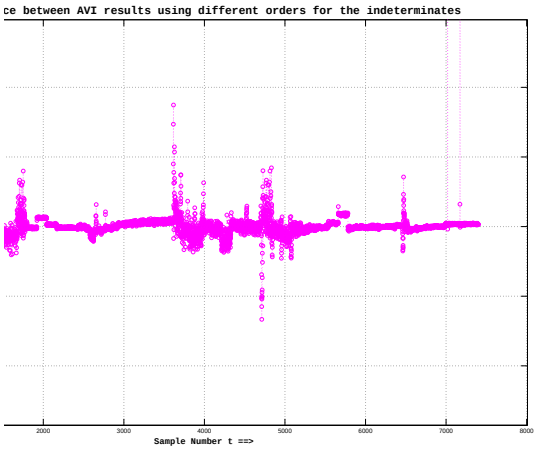


Fig. 1.8 Differences of the evaluations of two models.

Notice that also in g, the "last" indeterminate y_5 plays a dominant role. However, this time there is no physical inconsistency associated with this fact. Quite to the

contrary, the terms in the support of the model, and in particular the factors we put in parenthesis, have physical interpretations revealing the flow mechanisms inside the well. Although a detailed discussion of these interpretations would exceed the scope of this paper, it should be mentioned here that our findings have been assessed positively in discussions with production engineers and have been confirmed by dedicated field experiments.

There is, however, one aspect in this vein which warrants to be mentioned here. Recall the brief discussion in Section 1.B of the commonly accepted procedure to express the total production as a linear combination of the separate productions. The terms y_4y_5 and $y_4y_5^2$ in the above polynomial g indicate that g cannot be written as a linear combination of the separate productions which correspond to y_4 and y_5. Clearly, the inappropriate decomposition of the total production resulting from the traditional procedures may have substantial consequences for the production strategy used in the exploitation of the reservoir.

To wrap up the discussion, we note that the information about the two-zone well captured in the data set $\mathbb{X}$ has been coded in merely *20* functions, namely the terms in $\mathcal{O}$. Using suitable linear combinations of these terms, we can find excellent *estimators* of the oil production of the two-zone well under different production conditions. It should be stressed that such a physical interpretation can usually not be given for linear combinations of terms contained in the expression of g, nor to the individual monomials for that matter; in particular, their evaluations over $\mathbb{X}$ may be negative or exceed physically meaningful bounds. In this sense, the terms in $\mathcal{O}$ should be considered as purely *mathematical* states of the system into which the production of the well can be decomposed. The structure of this decomposition reveals deep insights into the production system which are only available via the described modeling procedure based on the AVI algorithm.

1.5 Stable Order Ideals

There is nothing so stable as change.
(Bob Dylan)

1.5.A. The SOI Algorithm. In this subsection we consider the following setting. Let $\mathbb{X} \subset \mathbb{R}^n$ be a finite set of points whose coordinates are known only with limited precision, and let

$$\mathcal{I}(\mathbb{X}) = \{f \in \mathbb{R}[x_1, \ldots, x_n] \mid f(p) = 0 \text{ for all } p \in \mathbb{X}\}$$

be its vanishing ideal. Our goal is to compare the different residue class rings $P/\mathcal{I}(\widetilde{\mathbb{X}})$ where $P = \mathbb{R}[x_1, \ldots, x_n]$ and $\widetilde{\mathbb{X}}$ is an admissible perturbation of $\mathbb{X}$, i.e. a set made up of points differing by less than the data uncertainty from the corresponding points of $\mathbb{X}$.

Given two distinct admissible perturbations $\widetilde{\mathbb{X}}_1$ and $\widetilde{\mathbb{X}}_2$ of $\mathbb{X}$, it can happen that their affine coordinate rings $P/\mathscr{I}(\widetilde{\mathbb{X}}_1)$ and $P/\mathscr{I}(\widetilde{\mathbb{X}}_2)$ as well as their vanishing ideals $\mathscr{I}(\widetilde{\mathbb{X}}_1)$ and $\mathscr{I}(\widetilde{\mathbb{X}}_2)$ have very different bases – this is a well known phenomenon in Gröbner basis theory. When dealing with a set $\mathbb{X}$ of empirical points, a notion of "numerically stable" basis of the quotient ring $P/\mathscr{I}(\mathbb{X})$ is necessary. A basis $\mathscr{O} \subseteq \mathbb{T}^n$ is stable if its residue classes form a vector space basis of $P/\mathscr{I}(\widetilde{\mathbb{X}})$ for any admissible perturbation $\widetilde{\mathbb{X}}$ of the empirical set $\mathbb{X}$. Furthermore, a stable order ideal $\mathscr{O}$ provides a common characterization of the ideals $\mathscr{I}(\mathbb{X})$ and $\mathscr{I}(\widetilde{\mathbb{X}})$ by means of their $\mathscr{O}$-border bases.

One way of dealing with the negative effects of data uncertainty is to replace elements of $\mathbb{X}$ which differ from each other by less than the data accuracy with a single representative point. This "preprocessing", using for instance the algorithms described in [1], may reduce the computational complexity but also loose information contained in the data. In general, it is not sufficient to eliminate the instabilities of exact bases of $P/\mathscr{I}(\mathbb{X})$. However, if we are given a finite set $\mathbb{X}$ of s well-separated empirical points, we can use the **S**table **O**rder **I**deal (SOI) Algorithm presented in this subsection. It computes a stable order ideal $\mathscr{O}$, and if $\mathscr{O}$ contains enough elements to form a basis of $P/\mathscr{I}(\mathbb{X})$, the corresponding stable border basis is also computed.

The following definition formalizes some concepts defined "empirically" in [30].

Definition 1.5.1. Let $p = (c_1, \ldots, c_n)$ be a point in $\mathbb{R}^n$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in (\mathbb{R}_+)^n$.

a) The pair (p, ε) is called an **empirical point** in $\mathbb{R}^n$. We shall denote it also by p^ε. The point p is called the **specific value** and ε is called the **tolerance** of p^ε.

b) A point $\widetilde{p} = (\widetilde{c}_1, \ldots, \widetilde{c}_n) \in \mathbb{R}^n$ is called an **admissible perturbation** of p if

$$\| ((\widetilde{c}_1 - c_1)/\varepsilon_1, \ldots, (\widetilde{c}_n - c_n)/\varepsilon_n) \| \leq 1$$

c) Let $\mathbb{X}^\varepsilon = \{p_1^\varepsilon, \ldots, p_s^\varepsilon\}$ be a set of empirical points which share the same tolerance ε, and let $\mathbb{X} = \{p_1, \ldots, p_s\}$ be its specific value. A set of points $\widetilde{\mathbb{X}} = \{\tilde{p}_1, \ldots, \tilde{p}_s\}$ is called an **admissible perturbation** of $\mathbb{X}$ if each point $\tilde{p}_i$ is an admissible perturbation of p_i.

d) Let a set $\mathbb{X}^\varepsilon = \{p_1^\varepsilon, \ldots, p_s^\varepsilon\}$ of empirical points be given with specific values $p_i = (c_{i1}, \ldots, c_{in})$. We introduce ns **error indeterminates**

$$\mathbf{e} = (e_{11}, \ldots, e_{s1}, e_{12}, \ldots, e_{s2}, \ldots, e_{1n}, \ldots, e_{sn})$$

Then the set $\widetilde{\mathbb{X}}(\mathbf{e}) = \{\widehat{p}_1, \ldots, \widehat{p}_s\}$ where $\widehat{p}_k = (c_{k1} + e_{k1}, \ldots, c_{kn} + e_{kn})$ is called the **generic perturbation** of $\mathbb{X}$.

Obviously, an admissible perturbation of $\mathbb{X}$ is obtained from the generic perturbation by substituting values $\tilde{e}_{ij}$ for the error indeterminates such that we have $\|(\tilde{e}_{i1}/\varepsilon_1, \ldots, \tilde{e}_{in}/\varepsilon_n)\| \leq 1$.

Next we define the notion of stability for order ideals.

Definition 1.5.2. Let $\mathscr{O}$ be an order ideal of $\mathbb{T}^n$. The set $\mathscr{O}$ is called **stable** w.r.t. $\mathbb{X}^\varepsilon$ if the evaluation matrix $\mathrm{eval}_{\widetilde{\mathbb{X}}}(\mathscr{O})$ has full rank for each admissible perturbation $\widetilde{\mathbb{X}}$ of $\mathbb{X}$. Furthermore, if $\mathscr{O}$ is also a basis of $P/\mathscr{I}(\mathbb{X})$, it is called a **stable quotient basis** of $\mathscr{I}(\mathbb{X})$.

Given any finite set of points $\mathbb{X}$, any quotient basis $\mathscr{O}$ for $\mathscr{I}(\mathbb{X})$ is stable w.r.t. $\mathbb{X}^\delta$ for a sufficiently small value of the tolerance δ. This is equivalent to saying that $\mathscr{O}$ has a "region of stability" w.r.t. $\mathbb{X}$ and follows from the next theorem.

Theorem 1.5.3. *Let $\mathbb{X}$ be a finite set of points in $\mathbb{R}^n$ and $\mathscr{O}$ a quotient basis for $\mathscr{I}(\mathbb{X})$. Then there exists a tolerance $\delta = (\delta_1, \ldots, \delta_n)$, with $\delta_i > 0$, such that $\mathscr{O}$ is stable w.r.t. $\mathbb{X}^\delta$.*

Proof. Let $\mathrm{eval}_{\mathbb{X}}(\mathscr{O})$ be the evaluation matrix of $\mathscr{O}$ at the points of $\mathbb{X}$. Its entries depend continuously on the points in $\mathbb{X}$. By hypothesis, the set $\mathscr{O}$ is a quotient basis for $\mathscr{I}(\mathbb{X})$. It follows that $\mathrm{eval}_{\mathbb{X}}(\mathscr{O})$ is invertible. Recalling that the determinant is a polynomial function in the matrix entries and noting that the entries of $\mathrm{eval}_{\mathbb{X}}(\mathscr{O})$ are polynomials in the points' coordinates, we can conclude that there exists a tolerance $\delta = (\delta_1, \ldots, \delta_n) \in (\mathbb{R}_+)^n$ such that $\det(\mathrm{eval}_{\widetilde{\mathbb{X}}}(\mathscr{O})) \neq 0$ for any perturbation $\widetilde{\mathbb{X}}$ of $\mathbb{X}$ in $\mathbb{X}^\delta$. □

Nevertheless, since the tolerance ε of the empirical points in $\mathbb{X}^\varepsilon$ is given *a priori* by the measurements, Theorem 1.5.3 does not provide a direct answer to the problem of stability. If the given tolerance ε for the points in $\mathbb{X}$ allows us to leave the "region of stability" of a chosen quotient basis $\mathscr{O}$, then $\mathscr{O}$ will not be stable w.r.t. $\mathbb{X}^\varepsilon$. Such a situation is shown in the following example.

Example 1.5.4. Let $\mathbb{X} = \{(0,2),\ (1,2.1),\ (2,1.9)\} \subseteq \mathbb{R}^2$ be a set of specified values. The order ideal $\mathscr{O} = \{1, y, y^2\}$ is a basis of $P/\mathscr{I}(\mathbb{X})$. Given the generic perturbation

$$\widetilde{\mathbb{X}}(\mathbf{e}) = \{(0+e_{11}, 2+e_{12}),\ (1+e_{21}, 2.1+e_{22}),\ (2+e_{31}, 1.9+e_{32})\}$$

the evaluation matrix of $\mathscr{O}$ at $\widetilde{\mathbb{X}}(\mathbf{e})$ is the Vandermonde matrix

$$\mathrm{eval}_{\widetilde{\mathbb{X}}(\mathbf{e})}(\mathscr{O}) = \begin{pmatrix} 1 & 2+e_{12} & (2+e_{12})^2 \\ 1 & 2.1+e_{22} & (2.1+e_{22})^2 \\ 1 & 1.9+e_{32} & (1.9+e_{32})^2 \end{pmatrix}$$

Since the matrix $\mathrm{eval}_{\widetilde{\mathbb{X}}}(\mathscr{O})$ is invertible if and only if the values $2+\tilde{e}_{12}$, $2.1+\tilde{e}_{22}$ and $1.9+\tilde{e}_{32}$ are pairwise distinct, we have that $\mathscr{O}$ is stable w.r.t. $\mathbb{X}^\varepsilon$ if the tolerance $\varepsilon = (\varepsilon_1, \varepsilon_2)$ satisfies $|\varepsilon_2| < 0.1$. Vice versa, if we consider $\mathbb{X}^{(\delta_1,\delta_2)}$, where $\delta_2 > 0.1$, there exists the admissible perturbation $\widetilde{\mathbb{X}} = \{(0,2),\ (1,2),\ (2,2)\}$ whose evaluation matrix $\mathrm{eval}_{\widetilde{\mathbb{X}}}(\mathscr{O})$ is singular. So, the order ideal $\mathscr{O}$ is not stable w.r.t. $\mathbb{X}^{(\delta_1,\delta_2)}$ since its "region of stability" is too small w.r.t. the given tolerance δ.

Intuitively, a border basis G of the vanishing ideal $\mathscr{I}(\mathbb{X})$ is considered to be structurally stable if, for each admissible perturbation $\widetilde{\mathbb{X}}$ of $\mathbb{X}$, it is possible to produce a border basis $\widetilde{G}$ of $\mathscr{I}(\widetilde{\mathbb{X}})$ only by means of a slight and continuous variation of the coefficients of the polynomials of G. This situation arises when G and $\widetilde{G}$ are founded on the same stable quotient basis $\mathscr{O}$, as shown in the following theorem (for a proof see [1]).

Theorem 1.5.5. *Let $\mathbb{X}^\varepsilon$ be a set of s distinct empirical points and $\mathscr{O} = \{t_1, \ldots, t_s\}$ a quotient basis for $\mathscr{I}(\mathbb{X})$ which is stable w.r.t. $\mathbb{X}^\varepsilon$. Then, for each admissible perturbation $\widetilde{\mathbb{X}}$ of $\mathbb{X}^\varepsilon$, the vanishing ideal $\mathscr{I}(\widetilde{\mathbb{X}})$ has an $\mathscr{O}$-border basis $\widetilde{G}$. Furthermore, if $\partial\mathscr{O} = \{b_1, \ldots, b_\nu\}$ is the border of $\mathscr{O}$ then $\widetilde{G}$ consists of ν polynomials of the form*

$$g_j = b_j - \sum_{i=1}^{s} \alpha_{ij} t_i \qquad \text{for } j \in \{1, \ldots, \nu\}$$

where the coefficients $a_{ij} \in \mathbb{R}$ satisfy the linear systems

$$\mathrm{eval}_{\widetilde{\mathbb{X}}}(b_j) = \sum_{i=1}^{s} \alpha_{ij} \mathrm{eval}_{\widetilde{\mathbb{X}}}(t_i)$$

Note that the coefficients α_{ij} of each polynomial $g_j \in \widetilde{G}$ are just the components of the solution α_j of the linear system $\mathrm{eval}_{\widetilde{\mathbb{X}}}(\mathscr{O})\,\alpha_j = \mathrm{eval}_{\widetilde{\mathbb{X}}}(b_j)$. It follows that the coefficients α_{ij} are continuous functions of the coordinates of the points of $\widetilde{\mathbb{X}}$. Since the order ideal $\mathscr{O}$ is stable w.r.t. $\mathbb{X}^\varepsilon$, they undergo only continuous variations as $\widetilde{\mathbb{X}}$ changes. Now the definition of stable border bases follows naturally.

Definition 1.5.6. Let $\mathbb{X}^\varepsilon$ be a finite set of distinct empirical points, and let $\mathscr{O}$ be a quotient basis for the vanishing ideal $\mathscr{I}(\mathbb{X})$. If $\mathscr{O}$ is stable w.r.t. $\mathbb{X}^\varepsilon$ then the $\mathscr{O}$-border basis G of $\mathscr{I}(\mathbb{X})$ is said to be **stable** w.r.t. the set $\mathbb{X}^\varepsilon$.

Given $\mathbb{X}$ and a stable quotient basis $\mathscr{O}$, it is possible to obtain a stable $\mathscr{O}$-border basis of $\mathscr{I}(\mathbb{X})$ by simple linear algebra computations. The SOI Algorithm addresses the problem of finding a stable quotient basis as follows. As in the Buchberger-Möller algorithm [6], the order ideal $\mathscr{O}$ is built stepwise: initially $\mathscr{O}$ comprises just the term 1; then at each iteration, a new term t is considered. If the evaluation matrix $\mathrm{eval}_{\widetilde{\mathbb{X}}}(\mathscr{O} \cup \{t\})$ has full rank for all admissible perturbations $\widetilde{\mathbb{X}}$ then t is added to $\mathscr{O}$; otherwise t is added to the corner set of the order ideal.

The rank condition is equivalent to checking whether $\rho(\widetilde{\mathbb{X}})$, the component of the evaluation vector $\mathrm{eval}_{\widetilde{\mathbb{X}}}(t)$ orthogonal to the column space of the matrix $\mathrm{eval}_{\widetilde{\mathbb{X}}}(\mathscr{O})$, vanishes for any admissible $\widetilde{\mathbb{X}}$. In the following theorem this check is greatly simplified by restricting it to first order error terms, as our interest is essentially focused on small perturbations $\widetilde{\mathbb{X}}$ of $\mathbb{X}$. In practice, the SOI algorithm solves an underdetermined system to test whether the first order approximation of $\rho(\widetilde{\mathbb{X}})$ vanishes for some admissible set $\widetilde{\mathbb{X}}$.

Theorem 1.5.7. (The Stable Order Ideal Algorithm (SOI))
Let $\mathbb{X}^\varepsilon = \{p_1^\varepsilon, \ldots, p_s^\varepsilon\}$ be a finite set of well-separated empirical points having a common tolerance $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. Let σ be a term ordering on $\mathbb{T}^n$ and $\gamma \geq 0$. Consider the following sequence of instructions.

S1 Start with the lists $\mathcal{O} = [1]$, $L = [x_1, \ldots, x_n]$, the empty list $C = [\,]$, the matrix $M_0 \in \mathrm{Mat}_{s\times 1}(\mathbb{R})$ with all entries equal to 1, and $M_1 \in \mathrm{Mat}_{s\times 1}(\mathbb{R})$ with all entries equal to 0.

S2 If $L = [\,]$ return the set $\mathcal{O}$ and stop. Otherwise, let $t = \min_\sigma(L)$ and delete it from L.

S3 Let v_0 and v_1 be the homogeneous components of degrees 0 and 1 of the evaluation vector $v = \mathrm{eval}_{\widetilde{\mathbb{X}}(\mathbf{e})}(t)$. Compute the vectors

$$\begin{aligned} \rho_0 &= v_0 - M_0\alpha_0 \\ \rho_1 &= v_1 - M_0\alpha_1 - M_1\alpha_0 \end{aligned}$$

where

$$\begin{aligned} \alpha_0 &= (M_0^{\mathrm{tr}}M_0)^{-1}M_0^{\mathrm{tr}}v_0 \\ \alpha_1 &= (M_0^{\mathrm{tr}}M_0)^{-1}(M_0^{\mathrm{tr}}v_1 + M_1^{\mathrm{tr}}v_0 - M_0^{\mathrm{tr}}M_1\alpha_0 - M_1^{\mathrm{tr}}M_0\alpha_0). \end{aligned}$$

S4 Let $C_t \in \mathrm{Mat}_{s,sn}(\mathbb{R})$ be such that $\rho_1 = C_t\mathbf{e}$. Let k be the maximum integer such that the matrix $\widehat{C}_t$, formed by selecting the first k rows of C_t, has minimum singular value $\widehat{\sigma}_k$ greater than $\|\varepsilon\|$. Let $\widehat{\rho}_0$ be the vector comprising the first k elements of ρ_0, and let $\widehat{C}_t^\dagger$ be the pseudoinverse of $\widehat{C}_t$. Compute $\widehat{\delta} = -\widehat{C}_t^\dagger\widehat{\rho}_0$, which is the minimal 2-norm solution of the underdetermined system $\widehat{C}_t\widehat{\delta} = -\widehat{\rho}_0$.

S5 If $\|\hat{\delta}\| > (1+\gamma)\sqrt{s}\|\varepsilon\|$, then adjoin the vector v_0 as a new column of M_0 and the vector v_1 as a new column of M_1. Append the power product t to $\mathcal{O}$, and add to L those elements of $\{x_1t, \ldots, x_nt\}$ which are not multiples of an element of L or C. Continue with step S2.

S6 Otherwise append t to the list C, and remove from L all multiples of t. Continue with step S2.

This is an algorithm which returns an order ideal $\mathcal{O} \subset \mathbb{T}^n$. If for every admissible perturbation $\widetilde{\mathbb{X}}$ the value γ satisfies $\|\rho_{2+}(\widetilde{\mathbb{X}})\| \leq \gamma\sqrt{s}\|\varepsilon\|^2$, where $\rho_{2+}(\widetilde{\mathbb{X}})$ is the evaluation at $\widetilde{\mathbb{X}}$ of the component of $\rho(\widetilde{\mathbb{X}}(\mathbf{e}))$ of degree greater than 1, then $\mathcal{O}$ is an order ideal which is stable w.r.t. the empirical set $\mathbb{X}^\varepsilon$. In particular, when $\#\mathcal{O} = s$, the ideal $\mathscr{I}(\mathbb{X})$ has a corresponding stable border basis w.r.t. $\mathbb{X}^\varepsilon$.

To implement the SOI Algorithm a value of γ has to be chosen even if an estimate of $\|\rho_{2+}(\widetilde{\mathbb{X}})\|$ is unknown. Since we consider small perturbations $\widetilde{\mathbb{X}}$ of the set $\mathbb{X}$, in most cases $\rho_0 + \rho_1(\widetilde{\mathbb{X}})$ is a good linear approximation of $\rho(\widetilde{\mathbb{X}})$. For this reason $\|\rho_{2+}(\widetilde{\mathbb{X}})\|$ is small and a value of $\gamma \ll 1$ can be chosen to obtain a set $\mathcal{O}$ which is stable w.r.t. $\mathbb{X}^\varepsilon$. On the other hand, if ρ is not well approximated by its

homogeneous components of degrees 0 and 1 the strategy of the SOI algorithm loses its meaning, since it is based on a first order analysis.

1.5.B. Comparison of the SOI and AVI Algorithms. Now we present some numerical examples to show the effectiveness of the SOI and AVI algorithms. The first two examples show how the algorithms detect simple geometrical configurations almost satisfied by the given set $\mathbb{X}$.

Example 1.5.8. (Four Almost Aligned Points)
Let $\mathbb{X}^\varepsilon$ be a set of empirical points with the specified values

$$\mathbb{X} = \{(0, 0.01),\ (0.34, 0.32),\ (0.65, 0.68),\ (0.99, 1)\} \subseteq \mathbb{R}^2$$

and the tolerance $\varepsilon = (0.03, 0.03)$.

a) The SOI algorithm computes the quotient basis $\mathscr{O} = \{1, y, y^2, y^3\}$ which is stable w.r.t. $\mathbb{X}^\varepsilon$. Hence we can compute the stable border basis G founded on it and get

$$G = \begin{cases} x - 0.654y^3 + 1.013y^2 - 1.362y + 0.014 \\ xy - 0.303y^3 - 0.552y^2 - 0.137y + 0.001 \\ xy^2 - 1.16y^3 + 0.238y^2 - 0.068y + 0.001 \\ xy^3 - 2.094y^3 + 1.368y^2 - 0.266y + 0.002 \\ y^4 - 2.01y^3 - 1.238y^2 - 0.23y + 0.002 \end{cases}$$

The algorithm also yields the almost vanishing polynomial $f = x - 0.984y$. This polynomial highlights the fact that $\mathbb{X}$ contains "almost aligned" points. Since the quotient basis $\mathscr{O}$ is stable w.r.t. $\mathbb{X}^\varepsilon$, we can conclude that there exists a small perturbation $\widetilde{\mathbb{X}}$ of $\mathbb{X}$ containing aligned points and for which the associated evaluation matrix $\mathrm{eval}_{\widetilde{\mathbb{X}}}(\mathscr{O})$ is invertible. Notice that this fact is not easily discernible from the computed border basis.
A further interesting polynomial is obtained by taking the difference of f and the first border basis polynomial. The resulting polynomial $h = 0.654y^3 - 1.013y^2 + 0.378y - 0.014$ has small values at the points of $\mathbb{X}$. This is not a contradiction to the "almost linear independence" of $\mathscr{O}$ in the sense of [2], since there is no admissible perturbation of $\mathbb{X}$ for which h vanishes. The correct interpretation is that there is an almost $\mathscr{O}$-border prebasis close to the computed $\mathscr{O}$-border basis which is not an approximate $\mathscr{O}$-border basis.

b) A completely different result is obtained by applying the Buchberger-Möller algorithm to the set $\mathbb{X}$. We use the same term ordering σ and obtain the following σ-Gröbner basis H of $\mathscr{I}(\mathbb{X})$:

$$\begin{array}{l} x^2 - 5525/5324y^2 - 30456/33275x + 103649/106480y - 6409/665500 \\ xy - 1358/1331y^2 - 15391/33275x + 32811/66550y - 8033/1663750 \\ y^3 - 205967/133100y^2 - 1271124/831875x + 1384811/665500y \\ \quad - 429556/20796875 \end{array}$$

The associated quotient basis is $\mathscr{O}_\sigma(\mathscr{I}(\mathbb{X})) = \mathbb{T}^2 \setminus \mathrm{LT}_\sigma\{\mathscr{I}(\mathbb{X})\} = \{1, y, x, y^2\}$. We observe that $\mathscr{O}_\sigma(\mathscr{I}(\mathbb{X}))$ is not stable because the matrix $\mathrm{eval}_{\widetilde{\mathbb{X}}}(\mathscr{O}_\sigma(\mathscr{I}(\mathbb{X})))$ is singular for some admissible perturbations of $\mathbb{X}$. In particular, the information that the points of $\mathbb{X}$ are "almost aligned" is not at all evident from H.

c) Finally, we apply the AVI algorithm to $\mathbb{X}$. If we use $\varepsilon = 0.05$, we obtain the quotient basis $\mathscr{O} = \{1, y, y^2\}$ and as the approximate $\mathscr{O}$-border basis

$$\begin{aligned} x &- 0.984y \\ xy &- 1.013y^2 + 0.03y - 0.004 \\ xy^2 &- 1.568y^2 + 0.614y - 0.026 \\ y^3 &- 1.556y^2 + 0.588y - 0.023 \end{aligned}$$

Notice that the first and last polynomial generate the vanishing ideal of the set of *three* points $\mathbb{X}' = \{(0.044, 0.044), (0.522, 0.529), (0.967, 0.982)\}$. Thus the "almost alignment" of $\mathbb{X}$ was correctly detected, and the algorithm found a cubic curve passing close to all four points.
If instead we apply the AVI algorithm with $\varepsilon = 0.03$, which is approximately the size of the data inaccuracies inherent in $\mathbb{X}$, we get $\mathscr{O} = \{1, y, y^2, y^3\}$ and the approximate border basis

$$\begin{aligned} x &- 0.984y \\ xy &- -1.013y^2 + 0.03y - 0.004 \\ xy^2 &- 1.16y^3 + 0.237y^2 - 0.068y \\ xy^3 &- 2.094y^3 + 1.367y^2 - 0.265y + 0.002 \\ y^4 &- 2.01y^3 + 1.237y^2 - 0.229y + 0.002 \end{aligned}$$

Here the ideal generated by the first and last polynomial corresponds to four perfectly aligned points very close to the points of $\mathbb{X}$.

In the following example we show the behavior of the SOI and AVI algorithms when applied to two sets of points with similar geometrical configuration but with different cardinality.

Example 1.5.9. (Points Close to a Circle)
Let $\mathbb{X}_8$ and $\mathbb{X}_{16}$ be sets of points created by perturbing slightly the coordinates of 8 and 16 points lying on the unit circle $x^2 + y^2 - 1 = 0$.

a) First we apply the SOI algorithm with tolerance $\varepsilon = (0.01, 0.01)$. The following table summarizes the results. The first two columns contain the name of the processed set and the value of its cardinality. The column labeled with "Corners" refers to the set of corners of the stable order ideal computed by the algorithm.
Note that the sets of corners of the stable quotient bases computed by the SOI algorithm always contain the power product x^2. This means that there is a numerical linear dependence among the empirical vectors associated to the power products $\{1, y, x, y^2, xy, x^2\}$ and that some useful information on the geometrical configuration of the points could be found.

Input	$\#\mathbb{X}_i$	$\#\mathcal{O}$ (SOI)	Corners (SOI)	$\#\mathcal{O}$ (AVI)	Corners (AVI)
$\mathbb{X}_8$	8	8	$\{x^2, xy^3, y^5\}$	8	$\{x^2, xy^3, y^5\}$
$\mathbb{X}_{16}$	16	16	$\{x^2, xy^7, y^9\}$	11	$\{x^2, xy^5, y^6\}$

Table 1.3 SOI and AVI on sets of points close to a circle

If we enlarge the tolerances, however, already for $\varepsilon = (0.04, 0.04)$ the SOI algorithm finds no stable border basis for X_{16} anymore.

b) Now we apply the AVI algorithm. Since the points are near the unit circle, no normalization is necessary. We use for instance $\varepsilon = 0.06$.
For $\mathbb{X}_8$, we obtain the same order ideal $\mathcal{O} = \{1, x, y, xy, y^2, xy^2, y^3, y^4\}$ as the SOI algorithm, and an approximate $\mathcal{O}$-border basis containing $\{0.57x^2 + 0.57y^2 - 0.57, 0.89xy^3 + 0.01y^3 - 0.44xy - 0.01y, 0.53y^5 - 0.79y^3 + 0.26y\}$. This shows that the circle close to the points has been detected.
Using $\mathbb{X}_{16}$, we find the order ideal $\mathcal{O} = \{1, x, y, xy, y^2, xy^2, y^3, xy^3, y^4\}$ and an approximate border basis which contains $0.57x^2 + 0.58y^2 - 0.57$. Again the close-by unit circle has been detected, but there are also three sextics passing close to the original 16 points. Unlike with the SOI algorithm, we find an approximate vanishing ideal of a smaller number (namely 11 instead of 16) of points here.

Our next example shows that the term ordering σ used in the SOI Algorithm is only an implementation detail. In general, any strategy that chooses the power product t such that $\mathcal{O} \cup \{t\}$ is always an order ideal can be applied. The example illustrates the case where σ can lead to an $\mathcal{O}$-border basis which does not contain the τ-Gröbner basis of $\mathcal{I}(\mathbb{X})$ for any term ordering τ. Similarly, the AVI algorithm can be modified in such a way that the same property holds.

Example 1.5.10. (A Quotient Basis Not of Gröbner Type)
Let $\mathbb{X}^\varepsilon$ be a set of distinct empirical points having

$$\mathbb{X} = \{(1,1),\ (0.82,-1),\ (-0.82,0.82),\ (-1,-0.82)\}$$

as the set of specified values and $\varepsilon = (0.1, 0.1)$ as the tolerance.

a) Applying the SOI algorithm to $\mathbb{X}^\varepsilon$, we get the quotient basis $\mathcal{O} = \{1, x, y, xy\}$ which is stable with respect to $\mathbb{X}^\varepsilon$. Let τ be any term ordering on $\mathbb{T}^n$ and $\mathcal{O}_\tau(\mathcal{I}(\mathbb{X})) = \mathbb{T}^n \backslash \mathrm{LT}_\tau\{\mathcal{I}(\mathbb{X})\}$ the quotient basis associated to τ. We note that we have $\mathcal{O} \neq \mathcal{O}_\tau(\mathcal{I}(\mathbb{X}))$ here. In fact, according to τ, either $x^2 <_\tau xy$ or $y^2 <_\tau xy$. Furthermore, at least one of the two evaluation vectors $\mathrm{eval}_\mathbb{X}(x^2)$, $\mathrm{eval}_\mathbb{X}(y^2)$ is linearly independent of $\{\mathrm{eval}_\mathbb{X}(1), \mathrm{eval}_\mathbb{X}(x), \mathrm{eval}_\mathbb{X}(y)\}$ so that one of x^2 or y^2 must belong to $\mathcal{O}_\tau(\mathcal{I}(\mathbb{X}))$. We conclude that the $\mathcal{O}$-border basis of $\mathcal{I}(\mathbb{X})$ does not contain any Gröbner basis of $\mathcal{I}(\mathbb{X})$.

b) Next we start from the set $\mathbb{X}$ and use the AVI algorithm with $\varepsilon = 0.1$ and $\varepsilon' = 0.01$. The result is the order ideal $\mathcal{O} = \{1, x, y, xy\}$ and the (uni-

tary) approximate border basis $G = \{0.76x^2 - 0.15xy - 0.62,\ 0.76y^2 - 0.13x - 0.63,\ 0.76x^2y - 0.12x - 0.62y,\ 0.76xy^2 - 0.63x - 0.11\}$.

Our final example illustrates that the order ideal $\mathscr{O}$ can have cardinality less than $s = \#\mathbb{X}$ both for the SOI and the AVI algorithm, but for different reasons. In the case of the SOI algorithm this happens when the tolerance on the points is, in some sense, too large. With a fixed set of specified values, the SOI algorithm may produce different results for different values of ε, some of which do not span all of $P/\mathscr{I}(\mathbb{X})$. For the AVI algorithm, the computed order ideal may satisfy $\#\mathscr{O} < s$ even for small ε. The reason is that the algorithm may detect many low degree polynomials vanishing ε-approximately at the given point, and those polynomials may generate a zero-dimensional ideal of lower codimension.

Example 1.5.11. (Five Points Close to Two Conics and a Cubic)
Let $\mathbb{X} = \{(0,1), (0.2,0.4), (0.28,0.28), (0.4,0.2), (1,0)\} \subset \mathbb{R}^2$.

a) First we apply the SOI algorithm to the set of well-separated empirical points $\mathbb{X}^\varepsilon$ with specified values $\mathbb{X}$ and tolerance $\varepsilon = (0.02, 0.02)$. We find the stable order ideal $\mathscr{O} = \{1, y, x, y^2\}$. However, this is not a quotient basis, so we cannot obtain the corresponding stable border basis. This is due to the fact that the points of $\mathbb{X}$ lie close to the hyperbola $xy + 0.17x + 0.14y - 0.17 = 0$, the ellipse $(x - 0.95)^2 + 0.87(y-1)^2 - 0.9 = 0$ and the cubic defined by the equation $y^3 - 1.8y^2 + 0.23x - 1.03y - 0.23 = 0$. So, if the tolerance ε is too big, they "almost satisfy" all of them.
Observe how the problem does not arise if we use a smaller tolerance, e.g. $\delta = (0.01, 0.01)$. Applying SOI to $\mathbb{X}^\delta$, we obtain the stable quotient basis $\mathscr{O}' = \{1, y, x, xy, y^2\}$ and its corresponding border basis

$$G' = \begin{cases} x^2 + 3.83xy + y^2 - 1.23x - 1.23y + 0.23 \\ y^3 - 0.07xy - 1.8y^2 + 0.22x + 1.02y - 0.22 \\ xy^2 - 0.07xy + 0.2y^2 - 0.05x - 0.25y + 0.05 \\ x^2y - 0.84xy - 0.2y^2 + 0.19y \end{cases}$$

b) Next we use the AVI algorithm. Choosing $\varepsilon = 0.06$, we get $\mathscr{O} = \{1, x, y\}$ and the (unitary) approximate border basis

$$G = \begin{cases} 0.52x^2 - 0.77x - 0.25y + 0.25 \\ 0.94xy + 0.18x + 0.18y - 0.18 \\ 0.51y^2 - 0.26x - 0.77y + 0.26 \end{cases}$$

The same result is produced for any $0.06 \leq \varepsilon \leq 0.25$. The set G approximates a system of generators of the vanishing ideal $\mathscr{I}(\widetilde{\mathbb{X}})$ of the point set $\widetilde{\mathbb{X}} = \{(0, 0.98), (0.28, 0.29), (0.98, 0)\}$. Notice that $\widetilde{\mathbb{X}}$ is approximately contained in all three conics.
A smaller choice of ε, for instance $\varepsilon = 0.01$, leads to $\mathscr{O}' = \{1, x, y, y^2\}$ and

$$G' = \begin{cases} 0.3x^2 + 0.3y^2 - 0.6x - 0.6y + 0.3 \\ 0.94xy + 0.18x + 0.18y - 0.18 \\ 0.95xy^2 + 0.19y^2 - 0.03x - 0.22y + 0.03 \\ 0.42y^3 - 0.77y^2 + 0.1x + 0.44y - 0.1 \end{cases}$$

The set G' approximates a system of generators of $\mathscr{I}(\widetilde{\mathbb{X}}')$ for $\widetilde{\mathbb{X}}' = \{(0,0.99), (0.21,0.37), (0.37,0.21), (0.99,0)\}$. Thus even this small choice of ε leads to a decrease in the codimension of the corresponding $\mathscr{I}(\mathbb{X})$.

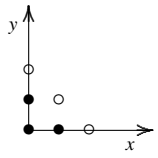
1.6 Border Basis and Gröbner Basis Schemes

Without geometry,
life is pointless.
(Sam Wormley)

Let $\mathscr{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal in $\mathbb{T}^n$. In this section we define a *moduli space*, called the border basis scheme, for *all* zero-dimensional ideals which have an $\mathscr{O}$-border basis. Then we define another space, called the Gröbner basis scheme, and explore their main properties, their connection to problems concerning approximate data, and their connection to Hilbert schemes of zero-dimensional schemes.

1.6.A. Two Basic Examples. Before starting with the technical details, we introduce two basic examples which will help us to understand the general theory.

Example 1.6.1. (Three Non-Collinear Points)
In this example we want to represent *all* zero-dimensional subschemes of $\mathbb{A}^2_K$ which share the property that the residue classes of the elements in $\mathscr{O} = \{1, x, y\}$ form a K-vector space basis of their coordinate ring. Another way of saying this is that we want to represent all ideals I in $P = K[x,y]$ such that the residue classes of the elements in $\mathscr{O}$ form a K-basis of P/I.



In this picture the elements of $\mathscr{O} = \{1, x, y\}$ are represented by bullets. Knowing that their residue classes form a K-vector space basis of P/I implies, in particular, that the elements represented by circles, i.e. x^2, xy, y^2 can be expressed modulo I as linear combinations of the elements in $\mathscr{O}$. In other words, the ideal I has to contain three polynomials of the form $g_1 = x^2 - c_{11} - c_{21}x - c_{31}y$, $g_2 = xy - c_{12} - c_{22}x - c_{32}y$, and $g_3 = y^2 - c_{13} - c_{23}x - c_{33}y$ for suitable values of the coefficients $c_{ij} \in K$.

But, of course, this is not the end of the discussion. For instance, the unit ideal contains such polynomials, but $\{1,x,y\}$ is not a basis modulo it. To achieve this property, we observe that $\{1,x,y\}$ is an order ideal of monomials and that the complementary monomial ideal is generated by $\{x^2,xy,y^2\}$. If σ is a degree-compatible term ordering, for instance $\sigma = \texttt{DegRevLex}$, we have $\mathrm{LT}_\sigma(g_1) = x^2$, $\mathrm{LT}_\sigma(g_2) = xy$, and $\mathrm{LT}_\sigma(g_3) = y^2$, independent of the values of the coefficients c_{ij}.

Macaulay's Basis Theorem (see [20], Theorem 1.5.7) implies that we have $\dim_K(P/I) = \dim_K(P/\mathrm{LT}_\sigma(I))$, and we want that this number is three. On the other hand, we have $\dim_K(P/(x^2,xy,y^2)) = 3$. Hence we want that $\mathrm{LT}_\sigma(I) = (x^2,xy,y^2)$. In other words, we want to impose that $\{g_1,g_2,g_3\}$ is a σ-Gröbner basis of I. Due to the particular shape of the equations involved, this requirement is equivalent to imposing that $\{g_1,g_2,g_3\}$ is the reduced σ-Gröbner basis of I.

How do we do that? There are two non-trivial fundamental syzygies of the tuple of terms (x^2,xy,y^2), namely $(-y,x,0)$ and $(0,-y,x)$. First we consider the S-polynomial $-yg_1 + xg_2 = c_{11}y + c_{21}xy + c_{31}y^2 - c_{12}x - c_{22}x^2 - c_{32}xy$. Using g_1,g_2,g_3, it can be rewritten as

$$\begin{aligned}(c_{21}c_{12} + c_{31}c_{13} - c_{22}c_{11} - c_{32}c_{12}) + (-c_{12} + c_{31}c_{23} - c_{32}c_{22})x \\ +(c_{11} + c_{21}c_{32} + c_{31}c_{33} - c_{22}c_{31} - c_{32}^2)y\end{aligned}$$

Second, we consider the S-polynomial $-yg_2 + xg_3 = c_{12}y + c_{22}xy + c_{32}y^2 - c_{13}x - c_{23}x^2 - c_{33}xy$. Using g_1,g_2,g_3, it can be rewritten as

$$\begin{aligned}(c_{22}c_{12} + c_{32}c_{13} - c_{23}c_{11} - c_{33}c_{12}) + (-c_{13} + c_{22}^2 + c_{32}c_{23} - c_{23}c_{21} - c_{33}c_{22})x \\ +(c_{12} + c_{22}c_{32} - c_{23}c_{31})y\end{aligned}$$

Imposing that $\{g_1,g_2,g_3\}$ is the reduced σ-Gröbner basis of I is therefore equivalent to imposing that the following expressions are all zero:

$$\begin{aligned}F_1 &= c_{21}c_{12} + c_{31}c_{13} - c_{22}c_{11} - c_{32}c_{12} \\ F_2 &= -c_{12} + c_{31}c_{23} - c_{32}c_{22} \\ F_3 &= c_{11} + c_{21}c_{32} + c_{31}c_{33} - c_{22}c_{31} - c_{32}^2 \\ F_4 &= c_{22}c_{12} + c_{32}c_{13} - c_{23}c_{11} - c_{33}c_{12} \\ F_5 &= -c_{13} + c_{22}^2 + c_{32}c_{23} - c_{23}c_{21} - c_{33}c_{22} \\ F_6 &= c_{12} + c_{22}c_{32} - c_{23}c_{31}\end{aligned}$$

Let J be the ideal of $K[c_{11},\ldots,c_{33}]$ generated by $\{F_1,F_2,F_3,F_4,F_5,F_6\}$. We note the equality $F_6 = -F_2 = c_{12} + c_{22}c_{32} - c_{23}c_{31}$ and check with CoCoA that $J = (F_2,F_3,F_5)$. By mapping c_{11} to $-c_{21}c_{32} - c_{31}c_{33} + c_{22}c_{31} + c_{32}^2$, c_{12} to $c_{31}c_{23} - c_{32}c_{22}$, and c_{13} to $c_{22}^2 + c_{32}c_{23} - c_{23}c_{21} - c_{33}c_{22}$, we define an isomorphism

$$K[c_{11},\ldots,c_{33}]/J \xrightarrow{\sim} K[c_{21},c_{31},c_{22},c_{32},c_{23},c_{33}]$$

The conclusion is that *all* zero-dimensional subschemes of $\mathbb{A}^2_K$ which have the property that the residue classes of the elements in $\{1,x,y\}$ form a K-vector space basis of their coordinate ring are parametrized by an affine space $\mathbb{A}^6_K$. Their vanishing ideals are generated by polynomials $\{g_1,g_2,g_3\}$ where

$$\begin{aligned} g_1 &= x^2-(-c_{21}c_{32}-c_{31}c_{33}+c_{22}c_{31}+c_{32}^2)-c_{21}x-c_{31}y\\ g_2 &= xy-(c_{31}c_{23}-c_{32}c_{22})-c_{22}x-c_{32}y\\ g_3 &= y^2-(c_{22}^2+c_{32}c_{23}-c_{23}c_{21}-c_{33}c_{22})-c_{23}x-c_{33}y \end{aligned}$$

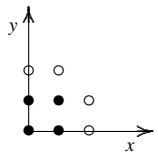
and the parameters which show up in these three polynomials can vary freely. Notice that this family contains only one monomial ideal, namely (x^2,xy,y^2).

To summarize this discussion, we note that we found the parametrizing scheme $\mathbb{A}^6_K$ by imposing that certain fundamental syzygies lift properly. This process is far from canonical. Nevertheless, it can be shown that the output is independent of the choices made (see [28], Proposition 3.5).

Furthermore, we observe that the dimension of the parameter space is six. How can we explain this number? We could argue as follows. Among the ideals represented by the family, there are the vanishing ideals of three non-collinear points. (For three collinear points, the set $\{1,x,y\}$ would not be linearly independent modulo their vanishing ideal.) Clearly, to represent three points in the affine plane one needs six independent coordinates. But we have to be careful: this argument does not work in general! Indeed, we cannot exclude *a priori* the existence of a component of the parameter space of higher dimension. In other words, we do not know *a priori* whether *degenerate schemes* which have $\{1,x,y\}$ as a basis of their coordinate ring can be represented as *limits* of sets of three distinct points. It turns out that this is true in our example, but not for more complicated order ideals $\mathcal{O}$. A similar counterintuitive situation arises in automatic theorem proving (see for instance [21], Section 6.7).

The next interesting case is the order ideal $\mathcal{O}=\{1,x,y,xy\}$ in $\mathbb{T}^2$.

Example 1.6.2. (Four Points)
As in the example before, we would like to parametrize *all* zero-dimensional subschemes of $\mathbb{A}^2_K$ such that the residue classes of the elements in $\mathcal{O}=\{1,x,y,xy\}$ form a K-vector space basis of their coordinate ring.



Let us try to argue as in the preceding example. The complement of the set $\mathcal{O}$ is the monomial ideal generated by $\{x^2,y^2\}$. Thus we want $\mathrm{LT}_\sigma(I)=(x^2,y^2)$. However, at this point we encounter a serious problem: no matter which term ordering we choose, it is not possible that both x^2 and y^2 are bigger than xy. The best we can do is to pick a degree-compatible term ordering, say $\sigma=$ `DegRevLex`,

and use two polynomials of the form $g_1 = x^2 - c_{11} - c_{21}x - c_{31}y - c_{41}xy$ and $g_2 = y^2 - c_{12} - c_{22}x - c_{32}y$. Then, for every choice of the parameters $c_{ij} \in K$, the set $\{g_1, g_2\}$ is the reduced σ-Gröbner basis of an ideal I such that the residue classes of the elements in $\{1, x, y, xy\}$ form a K-vector space basis of P/I.

Here we have seven free parameters. But four points in the affine plane need eight parameters to describe them completely. This shows that our Gröbner basis approach is not sufficient. A better way to proceed is to consider the border of the given order ideal. Its elements are marked by circles in the above picture. We represent every element in the border as a generic linear combination of $\{1, x, y, xy\}$ and impose that the given generic border prebasis is a border basis. In this way we obtain a set of equations which define an 8-dimensional moduli scheme (see Example 1.6.14).

1.6.B. Border Basis Schemes. The above examples indicate that border bases are well suited for describing families of affine subschemes of $\mathbb{A}^n$ whose coordinate rings have a given K-basis. In fact, they do the job better than Gröbner bases. It is time to provide the precise definitions and technical details necessary for the theoretical foundation of these observations.

Definition 1.6.3. Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal, let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$, and let $\{c_{ij} \mid 1 \le i \le \mu,\ 1 \le j \le \nu\}$ be a set of new indeterminates.

a) The **generic $\mathcal{O}$-border prebasis** is the set of polynomials $G = \{g_1, \ldots, g_\nu\}$ in $K[x_1, \ldots, x_n, c_{11}, \ldots, c_{\mu\nu}]$ given by

$$g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i \qquad \text{for } j = 1, \ldots, \nu$$

b) For $k = 1, \ldots, n$, let $\mathcal{A}_k \in \mathrm{Mat}_\mu(K[c_{ij}])$ be the k^{th} formal multiplication matrix associated to G (cf. [21], Def. 6.4.29). It is also called the k^{th} **generic multiplication matrix** with respect to $\mathcal{O}$.

c) The affine scheme $\mathbb{B}_\mathcal{O} \subseteq \mathbb{A}^{\mu\nu}$ defined by the ideal $I(\mathbb{B}_\mathcal{O})$ which is generated by the entries of the matrices $\mathcal{A}_k\mathcal{A}_\ell - \mathcal{A}_\ell\mathcal{A}_k$ with $1 \le k < \ell \le n$ is called the **$\mathcal{O}$-border basis scheme**.

d) The coordinate ring $K[c_{11}, \ldots, c_{\mu\nu}]/I(\mathbb{B}_\mathcal{O})$ of the scheme $\mathbb{B}_\mathcal{O}$ will be denoted by $B_\mathcal{O}$.

We observe that, by definition, the ideal $I(\mathbb{B}_\mathcal{O})$ is generated by polynomials of degree two. By [21], Thm. 6.4.30, a point $(\alpha_{ij}) \in K^{\mu\nu}$ yields a border basis $\sigma(G)$ when we apply the substitution $\sigma(c_{ij}) = \alpha_{ij}$ to G if and only if $\sigma(\mathcal{A}_k)\,\sigma(\mathcal{A}_\ell) = \sigma(\mathcal{A}_\ell)\,\sigma(\mathcal{A}_k)$ for $1 \le k < \ell \le n$. Therefore the K-rational points of $\mathbb{B}_\mathcal{O}$ are in 1–1 correspondence with the $\mathcal{O}$-border bases of zero-dimensional ideals in P, and thus with all zero-dimensional ideals having an $\mathcal{O}$-border basis.

In the following remark, we collect some basic properties of border basis schemes.

Remark 1.6.4. Let $\mathcal{O}$ be an order ideal in $\mathbb{T}^n$, and let $\mathbb{B}_\mathcal{O}$ be the $\mathcal{O}$-border basis scheme.

a) There is an irreducible component of $\mathbb{B}_{\mathcal{O}}$ of dimension $n\mu$ which is the closure of the set of radical ideals having an $\mathcal{O}$-border basis.
b) There is an example by Iarrobino (see [23] and [22], Example 5.6) which exhibits a border basis scheme having an irreducible component whose dimension is higher than $n\mu$.
c) For every term ordering σ, there is a subset of $\mathbb{B}_{\mathcal{O}}$ which parametrizes all ideals I such that $\mathcal{O} = \mathcal{O}_\sigma(I)$. These subsets have turned out to be useful for studying the Hilbert scheme parametrizing subschemes of $\mathbb{A}^n$ of length μ (see for instance [7] and [27]).
d) In the case $n = 2$ more precise information about $\mathbb{B}_{\mathcal{O}}$ is available: for instance, it is known that $\mathbb{B}_{\mathcal{O}}$ is reduced, irreducible and smooth of dimension 2μ (see [13], [15] and [24], Ch. 18).

Our next remark clarifies the relation between border basis schemes and Hilbert schemes.

Remark 1.6.5. For an order ideal $\mathcal{O}$ in $\mathbb{T}^n$, the border basis scheme $\mathbb{B}_{\mathcal{O}}$ can be embedded as an open affine subscheme of the Hilbert scheme parametrizing subschemes of $\mathbb{A}^n$ of length μ (see [24], Section 18.4). This can be seen as follows.

Let $I_{\mathcal{O}}$ be the monomial ideal generated by the complement of $\mathcal{O}$. The Hilbert polynomial of $P/I_{\mathcal{O}}$ is the constant polynomial μ. Among all schemes having this Hilbert polynomial there are the schemes for which $\mathcal{O}$ yields a basis of their coordinate ring. This condition defines a Zariski open subset.

As usual, a moduli space such as the border basis scheme comes together with a universal flat family. In the present setting it is defined as follows.

Definition 1.6.6. Let $G = \{g_1, \ldots, g_\nu\} \subset K[x_1, \ldots, x_n, c_{11}, \ldots, c_{\mu\nu}]$ with $g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$ for $j = 1, \ldots, \nu$ be the generic $\mathcal{O}$-border prebasis. We will denote the ring $K[x_1, \ldots, x_n, c_{11}, \ldots, c_{\mu\nu}]/(I(\mathbb{B}_{\mathcal{O}}) + (g_1, \ldots, g_\nu))$ by $U_{\mathcal{O}}$. Then the natural homomorphism of K-algebras

$$\Phi : B_{\mathcal{O}} \longrightarrow U_{\mathcal{O}} \cong B_{\mathcal{O}}[x_1, \ldots, x_n]/(g_1, \ldots, g_\nu)$$

is called the **universal $\mathcal{O}$-border basis family**.

What are the fibers of this family? It is easy to understand that they are the quotient rings P/I for which I is a zero-dimensional ideal which has an $\mathcal{O}$-border basis. The special fiber, i.e. the fiber corresponding to $(0, \ldots, 0)$, is the ring $P/(\partial\mathcal{O})$. It is the only fiber in the family which is defined by a monomial ideal. A remarkable result is the following.

Theorem 1.6.7. (The Universal Border Basis Family)
Let $\Phi : B_{\mathcal{O}} \longrightarrow U_{\mathcal{O}}$ be the universal $\mathcal{O}$-border basis family. Then the residue classes of the elements of $\mathcal{O}$ are a $B_{\mathcal{O}}$-module basis of $U_{\mathcal{O}}$. In particular, the map Φ is a flat homomorphism.

Proof. See [11] or [16]. For an elementary proof see [22], Theorem 3.4. □

Let us have a look at the first consequences of this fundamental result. A rational curve on the $\mathcal{O}$-border basis scheme corresponds to a surjective K-algebra homomorphism $\Psi : B_{\mathcal{O}} \longrightarrow K[z]$ of the corresponding affine coordinate rings. If we restrict the universal family of $\mathcal{O}$-border bases to this rational curve, we obtain the following flat deformation of border bases.

Corollary 1.6.8. *Let z be a new indeterminate, and let $\Psi : B_{\mathcal{O}} \longrightarrow K[z]$ be a surjective homomorphism of K-algebras. By applying the base change Ψ to the universal family Φ, we get a homomorphism of $K[z]$-algebras*

$$\Phi_{K[z]} = \Phi \otimes_{B_{\mathcal{O}}} K[z] : K[z] \longrightarrow U_{\mathcal{O}} \otimes_{B_{\mathcal{O}}} K[z]$$

Then the residue classes of the elements of $\mathcal{O}$ form a $K[z]$-module basis of the right-hand side. In particular, the map $\Phi_{K[z]}$ defines a flat family.

As explained in [22], this corollary can be used to construct flat deformations over $K[z]$ of border bases. Suppose the maximal ideal $\Psi^{-1}(z-1)$ corresponds to a given $\mathcal{O}$-border basis and the maximal ideal $\Psi^{-1}(z)$ is the ideal $(c_{11},\ldots,c_{\mu\nu})$ which corresponds to the border term ideal $(b_1,\ldots,b_\nu)$. In other words, suppose that there exists a rational curve which connects the given point to the point $(0,\ldots,0)$. Then the map $\Phi_{K[z]}$ defines a flat family over $K[z]$ whose generic fiber P/I is defined by the ideal I generated by the given $\mathcal{O}$-border basis and whose special fiber $P/(b_1,\ldots,b_\nu)$ is defined by the border term ideal.

In the next part of this subsection we try to construct explicit flat deformations to the border term ideal. The idea is to imitate the method used in Gröbner basis theory, namely the technique of homogenization. The first step is to deform to a suitable degree form ideal.

Lemma 1.6.9. *Let P be graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, let $\mathcal{O}$ be an order ideal in $\mathbb{T}^n$, and let $I \subset P$ be a homogeneous ideal which has an $\mathcal{O}$-border basis. Then this $\mathcal{O}$-border basis of I consists of homogeneous polynomials.*

Proof. See [22], Lemma 2.3. □

As for the idea to deform a border basis of I to a homogeneous border basis of the degree form ideal $\mathrm{DF}_W(I)$, we have the following result.

Theorem 1.6.10. (Deformation to the Degree Form Ideal)
Let $W = (w_1,\ldots,w_n) \in \mathrm{Mat}_{1,n}(\mathbb{N}_+)$ be a row of positive integers, let P be graded by W, and let $I \subset P$ be a zero-dimensional ideal. Then the following conditions are equivalent.

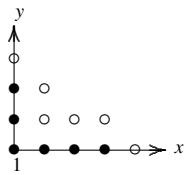
a) The ideal I has an $\mathcal{O}$-border basis, say $G = \{g_1,\ldots,g_\nu\}$, and we have $b_j \in \mathrm{Supp}(\mathrm{DF}_W(g_j))$ for $j = 1,\ldots,\nu$.
b) The degree form ideal $\mathrm{DF}_W(I)$ has an $\mathcal{O}$-border basis.

If these conditions are satisfied, the $\mathcal{O}$-border basis of $\mathrm{DF}_W(I)$ is $\mathrm{DF}_W(G) = \{\mathrm{DF}_W(g_1),\ldots,\mathrm{DF}_W(g_s)\}$ and there is a flat family $K[x_0] \longrightarrow \overline{P}/I\mathrm{hom}$ whose general fiber is isomorphic to P/I, where $I = (g_1,\ldots,g_\nu)$, and whose special fiber is isomorphic to $P/\mathrm{DF}_W(I)$, where $\mathrm{DF}_W(I) = (\mathrm{DF}_W(g_1),\ldots,\mathrm{DF}_W(g_\nu))$.

Proof. See [22], Theorem 2.4. □

Let us look at an example for the application of this theorem.

Example 1.6.11. Consider the ideal $I = (-2x^2+xy-y^2-1, 8y^3+10x+9y)$ in the polynomial ring $P = \mathbb{Q}[x,y]$. The degree form ideal of I with respect to the standard grading, i.e. the grading defined by $W = (1\ 1)$, is $\mathrm{DF}_W(I) = (-2x^2+xy-y^2, y^3)$. We want to use the order ideal $\mathcal{O} = \{1, x, x^2, x^3, y, y^2\}$ whose border is given by $\partial\mathcal{O} = \{xy, y^3, xy^2, x^2y, x^3y, x^4\}$.



It is easy to check that $\mathrm{DF}_W(I)$ has an $\mathcal{O}$-border basis, namely $H = \{h_1, \ldots, h_6\}$ with $h_1 = xy - 2x^2 - y^2$, $h_2 = y^3$, $h_3 = xy^2 + 4x^3$, $h_4 = x^2y + 2x^3$, $h_5 = x^3y$, and $h_6 = x^4$. Therefore the theorem says that I has an $\mathcal{O}$-border basis $G = \{g_1, \ldots, g_6\}$, and that $h_i = \mathrm{DF}_W(g_i)$ for $i = 1, \ldots, 6$. Indeed, if we compute this border basis we find that it is given by $g_1 = xy - 2x^2 - y^2 - 1$, $g_2 = y^3 + \frac{5}{4}x + \frac{9}{8}y$, $g_3 = xy^2 + 4x^3 + \frac{3}{4}x - \frac{1}{8}y$, $g_4 = x^2y + 2x^3 - \frac{1}{4}x - \frac{1}{8}y$, $g_5 = x^3y - \frac{1}{2}x^2 - \frac{1}{8}y^2 - \frac{3}{32}$, and $g_6 = x^4 - \frac{1}{64}$.

An easy modification of this example shows that the implication "a) $\Longrightarrow$ b)" in the theorem is not true without the hypothesis $b_j \in \mathrm{Supp}(\mathrm{DF}_W(g_j))$. This observation inspires the following definition.

Definition 1.6.12. Let P be graded by a matrix $W \in \mathrm{Mat}_{1,n}(\mathbb{N}_+)$. The order ideal $\mathcal{O}$ is said to have a **maxdeg$_W$ border** if $\deg_W(b_j) \geq \deg_W(t_i)$ for $i = 1, \ldots, \mu$ and $j = 1, \ldots, \nu$. In other words, no term in $\mathcal{O}$ is allowed to have a degree larger than any term in the border.

Using this notion, we can combine the deformation given by the theorem with a second deformation from the degree form ideal to the border term ideal by using the following result.

Theorem 1.6.13. (Homogeneous Maxdeg Border Bases)
Suppose that the order ideal $\mathcal{O}$ has a maxdeg_W *border. Let $I \subset P$ be a homogeneous ideal which has an $\mathcal{O}$-border basis $G = \{g_1, \ldots, g_\nu\}$. Then there exists a flat family $K[z] \longrightarrow K[z][x_1, \ldots, x_n]/J$ such that $\mathcal{O}$ is a $K[z]$-basis of the right-hand side, such that $J|_{z\mapsto 1} \cong I$, and such that $J|_{z\mapsto 0} \cong (b_1, \ldots, b_\nu)$. In fact, the ideal J may be defined by writing $g_j = b_j - \sum_{i=1}^{\mu} c_{ij}t_i$ and replacing $c_{ij} \in K$ by $c_{ij}z \in K[z]$ for all i, j.*

Proof. See [22], Theorem 5.3. □

To get a good grasp of these deformations, we look at one particular border basis scheme in detail, namely the one corresponding to Example 1.6.2.

Example 1.6.14. Consider the case $n = 2$ and $\mathcal{O} = \{1, x, y, xy\}$. The border of $\mathcal{O}$ is $\partial\mathcal{O} = \{y^2, x^2, xy^2, x^2y\}$, so that in our terminology we have $\mu = 4$, $\nu = 4$, $t_1 = 1$, $t_2 = x$, $t_3 = y$, $t_4 = xy$, $b_1 = y^2$, $b_2 = x^2$, $b_3 = xy^2$, and $b_4 = x^2y$.

The generic multiplication matrices are

$$\mathscr{A}_x = \begin{pmatrix} 0 & c_{12} & 0 & c_{14} \\ 1 & c_{22} & 0 & c_{24} \\ 0 & c_{32} & 0 & c_{34} \\ 0 & c_{42} & 1 & c_{44} \end{pmatrix} \quad \text{and} \quad \mathscr{A}_y = \begin{pmatrix} 0 & 0 & c_{11} & c_{13} \\ 0 & 0 & c_{21} & c_{23} \\ 1 & 0 & c_{31} & c_{33} \\ 0 & 1 & c_{41} & c_{43} \end{pmatrix}$$

When we compute the ideal generated by the entries of $\mathscr{A}_x\mathscr{A}_y - \mathscr{A}_y\mathscr{A}_x$ and simplify its system of generators, we see that the ideal $I(\mathbb{B}_{\mathcal{O}})$ is generated by

$$\begin{array}{l} \{c_{23}c_{41}c_{42} - c_{21}c_{42}c_{43} + c_{21}c_{44} + c_{11} - c_{23},\ -c_{21}c_{32} - c_{34}c_{41} + c_{33}, \\ c_{34}c_{41}c_{42} - c_{32}c_{41}c_{44} + c_{32}c_{43} + c_{12} - c_{34},\ -c_{21}c_{32} - c_{23}c_{42} + c_{24}, \\ -c_{23}c_{32}c_{41} + c_{21}c_{32}c_{43} - c_{21}c_{34} + c_{13},\ c_{21}c_{42} + c_{41}c_{44} + c_{31} - c_{43}, \\ -c_{21}c_{34}c_{42} + c_{21}c_{32}c_{44} - c_{23}c_{32} + c_{14},\ c_{32}c_{41} + c_{42}c_{43} + c_{22} - c_{44}\} \end{array}$$

Thus there are eight free indeterminates, namely c_{21}, c_{23}, c_{32}, c_{34}, c_{41}, c_{42}, c_{43}, and c_{44}, while the remaining indeterminates depend on the free ones by the polynomial expressions above. From this we conclude that the border basis scheme $\mathbb{B}_{\mathcal{O}}$ is an *affine cell* of the corresponding Hilbert scheme, i.e. an open subset which is isomorphic to an affine space.

Its coordinate ring is explicitly represented by the isomorphism

$$B_{\mathcal{O}} \xrightarrow{\sim} K[c_{21}, c_{23}, c_{32}, c_{34}, c_{41}, c_{42}, c_{43}, c_{44}]$$

given by

$$\begin{array}{rcl} c_{11} & \longmapsto & -c_{23}c_{41}c_{42} + c_{21}c_{42}c_{43} - c_{21}c_{44} + c_{23} \\ c_{12} & \longmapsto & -c_{34}c_{41}c_{42} + c_{32}c_{41}c_{44} - c_{32}c_{43} + c_{34} \\ c_{13} & \longmapsto & c_{23}c_{32}c_{41} - c_{21}c_{32}c_{43} + c_{21}c_{34} \\ c_{14} & \longmapsto & c_{21}c_{34}c_{42} - c_{21}c_{32}c_{44} + c_{23}c_{32} \\ c_{22} & \longmapsto & -c_{32}c_{41} - c_{42}c_{43} + c_{44} \\ c_{24} & \longmapsto & c_{21}c_{32} + c_{23}c_{42} \\ c_{31} & \longmapsto & -c_{21}c_{42} - c_{41}c_{44} + c_{43} \\ c_{33} & \longmapsto & c_{21}c_{32} + c_{34}c_{41} \end{array}$$

Hence we have $U_{\mathcal{O}} \cong K[x, y, c_{21}, c_{23}, c_{32}, c_{34}, c_{41}, c_{42}, c_{43}, c_{44}]/(\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3, \widetilde{g}_4)$ where

$$\begin{aligned}
\widetilde{g}_1 &= y^2 - (-c_{23}c_{41}c_{42} + c_{21}c_{42}c_{43} - c_{21}c_{44} + c_{23}) \\
&\quad -c_{21}x - (-c_{21}c_{42} - c_{41}c_{44} + c_{43})y - c_{41}xy, \\
\widetilde{g}_2 &= x^2 - (-c_{34}c_{41}c_{42} + c_{32}c_{41}c_{44} - c_{32}c_{43} + c_{34}) \\
&\quad -(-c_{32}c_{41} - c_{42}c_{43} + c_{44})x - c_{32}y - c_{42}xy, \\
\widetilde{g}_3 &= xy^2 - (c_{23}c_{32}c_{41} - c_{21}c_{32}c_{43} + c_{21}c_{34}) \\
&\quad -c_{23}x - (c_{21}c_{32} + c_{34}c_{41})y - c_{43}xy, \\
\widetilde{g}_4 &= x^2y - (c_{21}c_{34}c_{42} - c_{21}c_{32}c_{44} + c_{23}c_{32}) \\
&\quad -(c_{21}c_{32} + c_{23}c_{42})x - c_{34}y - c_{44}xy,
\end{aligned}$$

The ideal $(\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3, \widetilde{g}_4)$ is the defining ideal of the family of all subschemes of length four of the affine plane which have the property that their coordinate ring admits $\mathcal{O}$ as a vector space basis. Since the border basis scheme is isomorphic to an affine space in this case, we can connect every point to the point corresponding to (x^2, y^2) by a rational curve. Therefore every ideal in the family can be deformed by a flat deformation to the monomial ideal (x^2, y^2). Algebraically, it suffices to substitute each free indeterminate c_{ij} with zc_{ij} where z is a new indeterminate. This yields the K-algebra homomorphism

$$\Phi_{K[z]} : K[z] \longrightarrow K[x, y, z, c_{21}, c_{23}, c_{32}, c_{34}, c_{41}, c_{42}, c_{43}, c_{44}]/(\overline{g}_1, \overline{g}_2, \overline{g}_3, \overline{g}_4)$$

where

$$\begin{aligned}
\overline{g}_1 &= y^2 - (-z^3c_{23}c_{41}c_{42} + z^3c_{21}c_{42}c_{43} - z^2c_{21}c_{44} + zc_{23}) \\
&\quad -zc_{21}x - (-z^2c_{21}c_{42} - z^2c_{41}c_{44} + zc_{43})y - zc_{41}xy, \\
\overline{g}_2 &= x^2 - (-z^3c_{34}c_{41}c_{42} + z^3c_{32}c_{41}c_{44} - z^2c_{32}c_{43} + zc_{34}) \\
&\quad -(-z^2c_{32}c_{41} - z^2c_{42}c_{43} + zc_{44})x - zc_{32}y - zc_{42}xy, \\
\overline{g}_3 &= xy^2 - (z^3c_{23}c_{32}c_{41} - z^3c_{21}c_{32}c_{43} + z^2c_{21}c_{34}) \\
&\quad -zc_{23}x - (z^2c_{21}c_{32} + z^2c_{34}c_{41})y - zc_{43}xy, \\
\overline{g}_4 &= x^2y - (z^3c_{21}c_{34}c_{42} - z^3c_{21}c_{32}c_{44} + z^2c_{23}c_{32}) \\
&\quad -(z^2c_{21}c_{32} + z^2c_{23}c_{42})x - zc_{34}y - zc_{44}xy,
\end{aligned}$$

By Corollary 1.6.8, the homomorphism $\Phi_{K[z]}$ is flat. For every point on the border basis scheme, it induces a flat deformation from the corresponding coordinate ring P/I to $P/(\partial\mathcal{O})$ where the border term ideal is $(\partial\mathcal{O}) = (y^2, x^2, xy^2, x^2y) = (x^2, y^2)$.

Finally, we want to draw the connection between border basis schemes and the approximate border bases defined in Section 4.

Remark 1.6.15. Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal in $\mathbb{T}^n$ and $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ its border.

a) An approximate $\mathscr{O}$-border basis $G = \{g_1, \ldots, g_\nu\}$ with $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i$, as defined in Section 4, yields a point $(a_{ij}) \in \mathbb{R}^{\mu\nu}$ which is *close* to the $\mathscr{O}$-border basis scheme. In fact, in Definition 1.4.7 we required that the normal remainders of the S-polynomials of neighbor syzygies are small. This implies that the coefficients of these normal remainders are small, and those coefficients are precisely the evaluations of the defining equations of $\mathbb{B}_\mathscr{O}$ at (a_{ij}).
b) The AVI algorithm computes an approximate $\mathscr{O}$-border basis. As mentioned above, this corresponds to a point p close to the border basis scheme. Therefore it is natural to ask how one can find an exact $\mathscr{O}$-border basis defined over $\mathbb{Q}$, i.e. a rational point on $\mathbb{B}_\mathscr{O}$ which is close to p. This problem, called the *rational recovery problem*, will be addressed in [19].

1.6.C. Gröbner Basis Schemes. In the first subsection we tried to use the shape of a Gröbner basis in order to parametrize families of zero-dimensional ideals, but we encountered difficulties. Then we saw that border bases are more suited for this purpose. Now we return to the Gröbner basis approach and try to put it in relation to the border basis technique. To this end, we plan to define $(\mathscr{O}, \sigma)$-Gröbner basis schemes.

Before we start the discussion, some extra bits of notation are required. Given an order ideal $\mathscr{O} = \{t_1, \ldots, t_\mu\}$, the set of minimal generators of the monoideal $\mathbb{T}^n \setminus \mathscr{O}$ (which are also called the **corners** of $\mathscr{O}$) is denoted by $c\mathscr{O}$, and we let η be the cardinality of $c\mathscr{O}$. Since $c\mathscr{O} \subseteq \partial\mathscr{O}$, it follows that $\eta \leq \nu$. Without loss of generality, we label the elements in $\partial\mathscr{O}$ so that $c\mathscr{O} = \{b_1, \ldots, b_\eta\}$.

Next we let σ be a term ordering on $\mathbb{T}^n$. Recall that, for an ideal I in the polynomial ring P, we denote the order ideal $\mathbb{T}^n \setminus \mathrm{LT}_\sigma(I)$ by $\mathscr{O}_\sigma(I)$. Moreover, we denote by $S_{\mathscr{O},\sigma}$ the set $\{c_{ij} \in \{c_{11}, \ldots, c_{\mu\nu}\} \mid b_j >_\sigma t_i\}$, by $L_{\mathscr{O},\sigma}$ the ideal generated by $\{c_{11}, \ldots, c_{\mu\nu}\} \setminus S_{\mathscr{O},\sigma}$ in $K[c_{11}, \ldots, c_{\mu\nu}]$, by $S_{c\mathscr{O},\sigma}$ the intersection $S_{\mathscr{O},\sigma} \cap \{c_{11}, \ldots, c_{\mu\eta}\}$, and by $L_{c\mathscr{O},\sigma}$ the ideal generated by $\{c_{11}, \ldots, c_{\mu\eta}\} \setminus S_{c\mathscr{O},\sigma}$ in $K[c_{11}, \ldots, c_{\mu\eta}]$. Furthermore, we denote the cardinality of $S_{c\mathscr{O},\sigma}$ by $s(c\mathscr{O}, \sigma)$.

Definition 1.6.16. For $j = 1, \ldots, \nu$, we define a polnomial g_j^* by

$$g_j^* \;=\; b_j - \sum_{\{i \mid b_j >_\sigma t_i\}} c_{ij} t_i \;=\; b_j - \sum_{c_{ij} \in S_{\mathscr{O},\sigma} \cap \{c_{1j}, \ldots, c_{\mu j}\}} c_{ij} t_i$$

a) The set of polynomials $\{g_1^*, \ldots, g_\eta^*\}$ is called the **generic $(\mathscr{O}, \sigma)$-Gröbner prebasis**.
b) The ideal $\big(L_{\mathscr{O},\sigma} + I(\mathbb{B}_\mathscr{O})\big) \cap K[S_{c\mathscr{O},\sigma}]$ of $K[S_{c\mathscr{O},\sigma}]$ defines an affine subscheme of $\mathbb{A}^{s(c\mathscr{O},\sigma)}$ which will be denoted by $\mathbb{G}_{\mathscr{O},\sigma}$ and called the **$(\mathscr{O}, \sigma)$-Gröbner basis scheme**. Its defining ideal $\big(L_{\mathscr{O},\sigma} + I(\mathbb{B}_\mathscr{O})\big) \cap K[S_{c\mathscr{O},\sigma}]$ will be denoted by $I(\mathbb{G}_{\mathscr{O},\sigma})$ and its coordinate ring $K[S_{c\mathscr{O},\sigma}]/I(\mathbb{G}_{\mathscr{O},\sigma})$ by $G_{\mathscr{O},\sigma}$.

Notice that the polynomial g_j^* is obtained from g_j by setting all indeterminates in $L_{\mathscr{O},\sigma} \cap \{c_{1j}, \ldots, c_{\mu j}\}$ to zero.

What is the relation between Gröbner basis schemes and border basis schemes? Well, by now it should be clear that a Gröbner basis scheme is a closed subscheme of the corresponding border basis scheme.

Example 1.6.17. Let us examine the inclusion $c\mathcal{O} \subseteq \partial\mathcal{O}$. If $\mathcal{O} = \{1,x,y,xy\}$ then $c\mathcal{O} = \{x^2,y^2\}$ while $\partial\mathcal{O} = \{x^2,y^2,x^2y,xy^2\}$, so that $c\mathcal{O} \subset \partial\mathcal{O}$. On the other hand, if $\mathcal{O} = \{1,x,y\}$ then $c\mathcal{O} = \partial\mathcal{O} = \{x^2,xy,y^2\}$.

Returning to $\mathcal{O} = \{1,x,y,xy\}$ we have $t_1 = 1$, $t_2 = x$, $t_3 = y$, $t_4 = xy$, $b_1 = x^2$, $b_2 = y^2$, $b_3 = x^2y$, $b_4 = xy^2$. Let $\sigma = \mathtt{DegRevLex}$, so that $x >_\sigma y$. Then we get $L_{\mathcal{O},\sigma} = L_{c\mathcal{O},\sigma} = (c_{42})$, $g_1^* = g_1$, $g_2^* = y^2 - (c_{12} + c_{22}x + c_{32}y)$, $g_3^* = g_3$, and $g_4^* = g_4$.

Having introduced the Gröbner basis scheme, we define a naturally associated universal family. We recall that $K[x_1,\ldots,x_n,c_{11},\ldots,c_{\mu\nu}]/\big(I(\mathbb{B}_\mathcal{O}) + (g_1,\ldots,g_\nu)\big)$ was denoted by $U_\mathcal{O}$ in Definition 1.6.6, and the natural homomorphism of K-algebras $\Phi : B_\mathcal{O} \longrightarrow U_\mathcal{O}$ was called the universal $\mathcal{O}$-border basis family.

Definition 1.6.18. The ring $K[x_1,\ldots,x_n,S_{c\mathcal{O},\sigma}]/\big(I(\mathbb{G}_{\mathcal{O},\sigma}) + (g_1^*,\ldots,g_\eta^*)\big)$ will be denoted by $U_{\mathcal{O},\sigma}$.

a) The natural homomorphism of K-algebras $\Psi : G_{\mathcal{O},\sigma} \longrightarrow U_{\mathcal{O},\sigma}$ is called the **universal $(\mathcal{O},\sigma)$-Gröbner basis family**.
b) The induced homomorphism of K-algebras $B_\mathcal{O}/\overline{L}_{\mathcal{O},\sigma} \longrightarrow U_\mathcal{O}/\overline{L}_{\mathcal{O},\sigma}$ will be denoted by $\overline{\Phi}$.

The next result shows than Gröbner basis schemes have a very nice property which is not shared by some border basis schemes. To help the reader, we simply write $\mathbf{x}$ for $x_1,\ldots,x_n$ and $\mathbf{c}$ for $c_{11},\ldots c_{\mu\nu}$.

Theorem 1.6.19. *There exists a system W of positive weights on the elements of $S_{c\mathcal{O},\sigma}$, a system $\overline{W}$ of positive weights on the elements of $S_{\mathcal{O},\sigma}$, and a system V of positive weights on $\mathbf{x}$ such that the following conditions hold true.*

a) The system $\overline{W}$ is an extension of the system W.
b) The ideal $I(\mathbb{G}_{\mathcal{O},\sigma})$ in $K[S_{c\mathcal{O},\sigma}]$ is W-homogeneous.
c) The ideal $I(\mathbb{G}_{\mathcal{O},\sigma}) + (g_1^*,\ldots,g_\eta^*)$ in $K[\mathbf{x},S_{c\mathcal{O},\sigma}]$ is (V,W)-homogeneous.
d) The image of $I(\mathbb{B}_\mathcal{O})$ in $K[S_{\mathcal{O},\sigma}] \cong K[\mathbf{c}]/L_{\mathcal{O},\sigma}$ is $\overline{W}$-homogeneous.
e) The image of $I(\mathbb{B}_\mathcal{O}) + (g_1^*,\ldots,g_\nu^*)$ in the ring $K[\mathbf{x},S_{\mathcal{O},\sigma}] \cong K[\mathbf{x},\mathbf{c}]/L_{\mathcal{O},\sigma}$ is $(V,\overline{W})$-homogeneous.

Proof. See [28], Theorem 2.8.

In other words, this theorem says that a Gröbner basis scheme has an intrinsic graded structure. It follows that it is isomorphic to an affine space if and only if the point corresponding to the unique monomial ideal is smooth (see [28], Corollary 3.7). Moreover, Gröbner basis schemes are connected. The analogous result for border basis schemes is not known. (A partial result is that if $\mathcal{O}$ has a maxdeg_W border, then $\mathbb{B}_\mathcal{O}$ is connected. This follows by combining Theorem 1.6.10 with [22], Theorem 5.3.)

Our next goal is to make the connection between Gröbner and border basis schemes more explicit. We recall the equality $I(\mathbb{G}_{\mathcal{O}}) = \big(L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}})\big) \cap K[S_{c\mathcal{O},\sigma}]$ which yields the homomorphism φ below. A further homomorphism ϑ is obtained as follows: let $\Theta : K[\mathbf{x}, S_{c\mathcal{O},\sigma}] \longrightarrow K[\mathbf{x},\mathbf{c}]$ be the natural inclusion of polynomial rings. Then clearly $I(\mathbb{G}_{\mathcal{O},\sigma}) + (g_1^*,\ldots,g_\eta^*) \subseteq \Theta^{-1}\big(L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}}) + (g_1,\ldots g_\nu)\big)$.

Now we consider the following commutative diagram of canonical homomorphisms.

$$\begin{array}{ccc} G_{\mathcal{O},\sigma} & \xrightarrow{\varphi} & B_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma} \\ \big\downarrow \Psi & & \big\downarrow \overline{\Phi} \\ U_{\mathcal{O},\sigma} & \xrightarrow{\vartheta} & U_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma} \end{array}$$

Using explicit representations, this diagram has the following form.

$$\begin{array}{ccc} K[S_{c\mathcal{O},\sigma}]/I(\mathbb{G}_{\mathcal{O},\sigma}) & \xrightarrow{\varphi} & K[\mathbf{c}]/\big(L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}})\big) \\ \big\downarrow \Psi & & \big\downarrow \overline{\Phi} \\ K[\mathbf{x}, S_{c\mathcal{O},\sigma}]/\big(I(\mathbb{G}_{\mathcal{O},\sigma}) + (g_1^*,\ldots,g_\eta^*)\big) & \xrightarrow{\vartheta} & K[\mathbf{x},\mathbf{c}]/\big(L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}}) + (g_1,\ldots g_\nu)\big) \end{array}$$

At this point we are ready for the following fundamental results about Gröbner basis schemes.

Theorem 1.6.20. (Gröbner Basis and Border Basis Schemes)
Let $\mathcal{O} = \{t_1,\ldots,t_\mu\}$ be an order ideal of monomials, and let σ be a term ordering on $\mathbb{T}^n$.

a) The classes of the elements in $\mathcal{O}$ form a $B_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}$-module basis of $U_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}$.
b) The classes of the elements in $\mathcal{O}$ form a $G_{\mathcal{O},\sigma}$-module basis of $U_{\mathcal{O},\sigma}$.
c) We have $I(\mathbb{G}_{\mathcal{O},\sigma}) + (g_1^*,\ldots,g_\eta^*) = \vartheta^{-1}\big(L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}}) + (g_1,\ldots g_\nu)\big)$.
d) The maps φ and ϑ in the above diagram are isomorphisms.

Proof. See [28], Theorem 2.9.

Corollary 1.6.21. *Let $\mathcal{O} = \{t_1,\ldots,t_\mu\}$ be an order ideal of monomials in P and let σ be a term ordering on $\mathbb{T}^n$.*

a) The affine scheme $\mathbb{G}_{\mathcal{O},\sigma}$ parametrizes all zero-dimensional ideals I in P for which $\mathcal{O} = \mathcal{O}_\sigma(I)$.
b) The fibers over the K-rational points of the universal $(\mathcal{O},\sigma)$ Gröbner family $\Psi : G_{\mathcal{O},\sigma} \longrightarrow U_{\mathcal{O},\sigma}$ are the quotient rings P/I for which I is a zero-dimensional ideal with the property that $\mathcal{O} = \mathcal{O}_\sigma(I)$. Moreover, the reduced σ-Gröbner basis of I is obtained by specializing the $(\mathcal{O},\sigma)$-Gröbner prebasis $\{g_1^*,\ldots,g_\eta^*\}$ to the corresponding maximal linear ideal.

Proof. See [28]. Corollary 2.11.

Finally, we reformulate these results in the language of algebraic geometry.

Remark 1.6.22. There is a commutative diagram

$$\begin{array}{ccc} \mathbb{G}_{\mathscr{O},\sigma} & \cong & \operatorname{Spec}(B_{\mathscr{O}}/\overline{L}_{\mathscr{O},\sigma}) \\ \uparrow{\scriptstyle \pi_\Psi} & & \uparrow{\scriptstyle \pi_{\overline{\Phi}}} \\ \operatorname{Spec}(U_{\mathscr{O},\sigma}) & \cong & \operatorname{Spec}(U_{\mathscr{O}}/\overline{L}_{\mathscr{O},\sigma}) \end{array}$$

of affine schemes, but more can be said. Let W, $\overline{W}$, and V be systems of positive weights, chosen suitably to satisfy Theorem 1.6.19. Then $G_{\mathscr{O},\sigma}$ is a W-graded ring, $B_{\mathscr{O}}$ is a $\overline{W}$-graded ring, $U_{\mathscr{O},\sigma}$ is a (V,W)-graded ring, and $U_{\mathscr{O}}/\overline{L}_{\mathscr{O},\sigma}$ is a $(V,\overline{W})$-graded ring.

Hence we see that the above diagram gives rise to a diagram

$$\begin{array}{ccc} \operatorname{Proj}(G_{\mathscr{O},\sigma}) & \cong & \operatorname{Proj}(B_{\mathscr{O}}/\overline{L}_{\mathscr{O},\sigma}) \\ \uparrow{\scriptstyle \Pi_\Psi} & & \uparrow{\scriptstyle \Pi_{\overline{\Phi}}} \\ \operatorname{Proj}(U_{\mathscr{O},\sigma}) & \cong & \operatorname{Proj}(U_{\mathscr{O}}/\overline{L}_{\mathscr{O},\sigma}) \end{array}$$

of projective schemes such that $\operatorname{Proj}(G_{\mathscr{O},\sigma}) \subset \mathbb{P}(W)$, $\operatorname{Proj}(B_{\mathscr{O}}/\overline{L}_{\mathscr{O},\sigma}) \subset \mathbb{P}(\overline{W})$, $\operatorname{Proj}(U_{\mathscr{O},\sigma}) \subset \mathbb{P}(V,W)$, and $\operatorname{Proj}(U_{\mathscr{O}}/\overline{L}_{\mathscr{O},\sigma}) \subset \mathbb{P}(V,\overline{W})$. The corresponding weighted projective spaces are denoted by $\mathbb{P}(W)$, $\mathbb{P}(\overline{W})$, $\mathbb{P}(V,W)$, and $\mathbb{P}(V,\overline{W})$.

Moreover, let $\mathbf{p} = (a_{ij}) \in \mathbb{G}_{\mathscr{O},\sigma}$ be a rational point, let $I \subset P$ be the corresponding ideal according to Corollary 1.6.21, let $v_i = \deg(x_i)$ in the V-grading, and let $w_{ij} = \deg(c_{ij})$ in the W-grading. Then it is well-known that the substitution $a_{ij} \longrightarrow t^{w_{ij}} a_{ij}$ gives rise to a flat family of ideals whose general fibers are ideals isomorphic to I, and whose special fiber is the monomial ideal $\operatorname{LT}_\sigma(I)$. In the setting of the first diagram, the rational monomial curve which parametrizes this family is a curve in $\mathbb{G}_{\mathscr{O},\sigma}$ which connects the two points representing I and $\operatorname{LT}_\sigma(I)$. In the setting of the second diagram, the rational monomial curve is simply a point in $\operatorname{Proj}(G_{\mathscr{O},\sigma}) \subset \mathbb{P}(W)$, which represents all of these ideals except for the special one.

In [28], Section 3, the relation between the construction of $I(\mathbb{G}_{\mathscr{O}})$ and other constructions described in the literature (see for instance [7] and [27]) is discussed. Our next remark collects the main points.

Remark 1.6.23. Starting with the generic σ-Gröbner prebasis $\{g_1^*, \ldots, g_\eta^*\}$, one can construct an affine subscheme of $\mathbb{A}^{s(c\mathscr{O},\sigma)}$ in the following way. As in the Buchberger Algorithm, one reduces the critical pairs of the leading terms of the σ-Gröbner prebasis as much as possible. The reduction stops when a polynomial is obtained which is a linear combination of the elements in $\mathscr{O}$ with coefficients in $K[S_{c\mathscr{O},\sigma}]$. Collecting all coefficients obtained in this way for all the critical pairs, one gets a set of polynomials which generates an ideal J in $K[S_{c\mathscr{O},\sigma}]$. Clearly, every

zero of J gives rise to a specialization of the generic σ-Gröbner prebasis which is, by construction, the reduced σ-Gröbner basis of a zero-dimensional ideal I in P for which we have $\mathcal{O} = \mathcal{O}_\sigma(I)$.

However, this procedure is not canonical, since for instance the choice of the critical pairs to be reduced and the order of the reduction steps is not fixed. Based on the construction presented in this subsection, one can show that all possible ideals J define *the same scheme*, namely the one defined in Definition 1.6.16.

Another interesting problem is to look for conditions under which the two schemes $G_{\mathcal{O},\sigma}$ and $B_{\mathcal{O}}$ are isomorphic. Proposition 3.11 of [28] yields a partial answer. Essentially, it says the following.

Proposition 1.6.24. *Let $\mathcal{O}$ be an order ideal and σ a term ordering on $\mathbb{T}^n$, and assume that the order ideal $\mathcal{O}$ is a* **σ-cornercut***, i.e. that we have $b >_\sigma t$ for every $b \in c\mathcal{O}$ and every $t \in \mathcal{O}$. Then the canonical embedding of $K[S_{\mathcal{O},\sigma}]$ into $K[c_{11}, \ldots, c_{\mu\nu}]$ induces an isomorphism between $G_{\mathcal{O},\sigma}$ and $B_{\mathcal{O}}$.*

The study of border basis and Gröbner basis schemes is still in its infancy. There are many open questions, for instance whether the converse of the preceding proposition holds, or whether border basis schemes are always connected. Although our journey from oil fields to Hilbert schemes ends here, the topics we discussed offer many possibilities to continue it in various directions.

When you have completed 95 percent of your journey,
you are only halfway there.
(Japanese proverb)

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