DEFORMATIONS OF BORDER BASES

MARTIN KREUZER AND LORENZO ROBBIANO

Abstract. Border bases have recently attracted a lot of attention. Here we study the problem of generalizing one of the main tools of Gröbner basis theory, namely the flat deformation to the leading term ideal, to the border basis setting. After showing that the straightforward approach based on the deformation to the degree form ideal works only under additional hypotheses, we introduce border basis schemes and universal border basis families. With their help the problem can be rephrased as the search for a certain rational curve on a border basis scheme. We construct the system of generators of the vanishing ideal of the border basis scheme in different ways and study the question of how to minimalize it. For homogeneous ideals, we also introduce a homogeneous border basis scheme and prove that it is an affine space in certain cases. In these cases it is then easy to write down the desired deformations explicitly.

1. Introduction

Let I be a zero-dimensional ideal in a polynomial ring $P = K[x_1, \ldots, x_n]$ over a field K, and let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal, i.e. a finite set of power products in P which is closed under taking divisors. An $\mathcal{O}$-*border basis* of I is a set of polynomials $G = \{g_1, \ldots, g_\nu\}$ of the form $g_j = b_j - \sum_{i=1}^{\mu} c_{ji} t_i$, where $\{b_1, \ldots, b_\nu\}$ is the border $\partial\mathcal{O} = (x_1\mathcal{O} \cup \cdots \cup x_n\mathcal{O}) \setminus \mathcal{O}$ of $\mathcal{O}$ and $c_{ji} \in K$, such that I is generated by G and $\mathcal{O}$ is a K-vector space basis of P/I. In recent years border bases have received considerable attention (see for instance [13], [14], [15], [20], and [22]). This is due to several reasons.

(1) Border bases generalize Gröbner bases: if one takes for $\mathcal{O}$ the complement of a leading term ideal of I with respect to some term ordering σ, the corresponding border basis contains the reduced σ-Gröbner basis of I.
(2) Border bases are more suitable for dealing with computations arising from real world problems. They are more stable with respect to small variations in the coefficients of the polynomials generating I and permit symbolic computations with polynomial systems having approximate coefficients (see for instance [1], [8], and [22]).
(3) Border bases are in general much more numerous than reduced Gröbner bases. For instance, if the given ideal I is invariant under the action of a group of symmetries, it is sometimes possible to find a border basis having these symmetries, but not a Gröbner basis.

The starting point for this paper is our attempt to generalize one of the fundamental results of Gröbner basis theory to the border basis setting, namely the fact that there exists a flat deformation from I to its leading term ideal $\mathrm{LT}_\sigma(I)$. More

Date: January 18, 2008.
2000 *Mathematics Subject Classification.* Primary 13P10; secondary 14D20,13D10, 14D15.
Key words and phrases. Border basis, Gröbner basis, deformation, Hilbert scheme.

precisely, we are looking at the following result. (Here and in the following we use the notation introduced in [16] and [17].)

Given a term ordering σ, the ring P can be graded by a row of positive integers $W = (w_1 \ \cdots \ w_n)$, i.e. by letting $\deg_W(x_i) = w_i$, such that the leading term ideal $\mathrm{LT}_\sigma(I)$ equals the degree form ideal $\mathrm{DF}_W(I)$. Using a homogenizing indeterminate x_0 and the grading of $\overline{P} = K[x_0, \dots, x_n]$ given by $\overline{W} = (1 \ w_1 \ \cdots \ w_n)$, the canonical K-algebra homomorphism $\Phi : K[x_0] \longrightarrow \overline{P}/I^{\mathrm{hom}}$ satisfies

(1) The ring $\overline{P}/I^{\mathrm{hom}}$ is a free $K[x_0]$-module.
(2) There are isomorphisms of K-algebras $\overline{P}/(I^{\mathrm{hom}} + (x_0)) \cong P/\mathrm{DF}_W(I)$ and $\overline{P}/(I^{\mathrm{hom}} + (x_0 - c)) \cong P/I$ for every $c \in K \setminus \{0\}$.

We express this by saying that there is a flat deformation from I to $\mathrm{DF}_W(I)$, and thus to $\mathrm{LT}_\sigma(I)$. In geometric jargon, we can say that, in the Hilbert scheme parametrizing affine schemes of length $\dim_K(P/I)$, the affine scheme defined by I is connected to the scheme defined by $\mathrm{DF}_W(I)$ via a rational curve parametrized by x_0. Thus the starting point for this paper is the question whether there exists a flat deformation from a zero-dimensional ideal I given by an $\mathcal{O}$-border basis $G = \{g_1, \dots, g_\nu\}$ as above to its border term ideal $\mathrm{BT}_\mathcal{O} = (b_1, \dots, b_\nu)$.

The direct approach taken in Section 2 is to try to imitate Gröbner basis theory and to use the flat deformation to the degree form ideal we just recalled. Unfortunately, this approach does not succeed in all cases, but only under the additional assumption that $\mathcal{O}$ has a maxdeg_W border, i.e. that no term in $\mathcal{O}$ has a larger degree than a term in the border $\partial\mathcal{O}$.

Therefore it is necessary to dig deeper into the problem and find other ways of constructing the desired flat deformations. In Section 3 we take a step back and view the task from a more global perspective. All zero-dimensional ideals having an $\mathcal{O}$-border basis can be parametrized by a scheme $\mathbb{B}_\mathcal{O}$ which we call the *$\mathcal{O}$-border basis scheme*. Using the condition that the generic multiplication matrices have to commute, we give explicit equations defining $\mathbb{B}_\mathcal{O}$ in a suitable affine space (see Definition 3.1).

A moduli space such as the border basis scheme usually comes together with a universal family: this is a morphism from $\mathbb{B}_\mathcal{O}$ to another scheme whose fibers are precisely the schemes defined by the ideals having an $\mathcal{O}$-border basis. The fundamental result about this *universal border basis family* is that it is flat. In fact, in Theorem 3.4 we give an elementary, explicit proof that $\mathcal{O}$ is a basis for the entire family, viewed as a module over the coordinate ring of the border basis scheme. Hence the construction of the desired flat deformation of an ideal to its border term ideal is equivalent to finding suitable rational curves on the border basis scheme (see Corollary 3.5).

To examine the border basis scheme further, we have a more detailed look at the system of generators of its vanishing ideal in Section 4. The technique of lifting neighbor syzygies (introduced in [13] and [22], and independently in [10]) provides us with a different way of constructing a system of generators of $I(\mathbb{B}_\mathcal{O})$ (see Proposition 4.1). Using suitable examples, including the well-known Example 4.2 of a singularity on a Hilbert scheme, we disprove several claims in [22] with respect to the possibility of removing redundant generators from this system. On the positive side, in Proposition 4.5 we provide a criterion for eliminating some unnecessary generators.

The final Section 5 introduces the homogeneous border basis scheme $\mathbb{B}_{\mathcal{O}}^{\hom}$. It parametrizes all homogeneous zero-dimensional ideals having an $\mathcal{O}$-border basis and is obtained from the border basis scheme by intersecting it with a suitable linear space. Our main result about $\mathbb{B}_{\mathcal{O}}^{\hom}$ is that it is an affine space (and not only isomorphic to an affine space) if $\mathcal{O}$ has a maxdeg_W border (see Theorem 5.3). This theorem is a nice tool which can be employed to produce good deformations (see Example 5.4) and to recreate the construction of reducible Hilbert schemes (see Example 5.6).

Here we close this introduction by pointing out that all computations were done using the computer algebra system CoCoA(see [3]) and that even great artists can be too pessimistic at times.

Deformations simply do not exist.
(Pablo Picasso)

2. Deformation to the Border Form Ideal

One of the fundamental results of Gröbner basis theory is that there exists a flat deformation of a polynomial ideal to its leading term ideal. This deformation is achieved by taking a Gröbner basis of the ideal, viewing it as a Macaulay basis with respect to a suitably chosen $\mathbb{N}$-grading, homogenizing it, and letting the homogenizing indeterminate tend to zero. An analogous fact for border bases of zero-dimensional polynomial ideals is not known in general. In this section we shall prove some partial results in this direction.

In the following we let K be a field, $P = K[x_1, \dots, x_n]$ a polynomial ring, and $I \subset P$ a zero-dimensional ideal. Recall that an *order ideal* $\mathcal{O}$ is a finite set of terms in $\mathbb{T}^n = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_i \geq 0\}$ such that all divisors of a term in $\mathcal{O}$ are also contained in $\mathcal{O}$. The set $\partial\mathcal{O} = (x_1\mathcal{O} \cup \cdots \cup x_n\mathcal{O}) \setminus \mathcal{O}$ is called the *border* of $\mathcal{O}$. By repeating this construction, we define the *higher borders* $\partial^i\mathcal{O}$ for $i \geq 1$ and we let $\partial^0\mathcal{O} = \mathcal{O}$. The number $\mathrm{ind}_{\mathcal{O}}(t) = \min\{i \geq 0 \mid t \in \partial^i\mathcal{O}\}$ is called the *$\mathcal{O}$-index* of a term $t \in \mathbb{T}^n$.

Definition 2.1. Let $\mathcal{O} = \{t_1, \dots, t_\mu\}$ be an order ideal and $\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$ its border.

a) A set of polynomials $G = \{g_1, \dots, g_\nu\} \subseteq I$ is called an *$\mathcal{O}$-border prebasis* of I if it is of the form $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i$ with $a_{ij} \in K$.
b) An $\mathcal{O}$-border prebasis of I is called an *$\mathcal{O}$-border basis* of I if $P = I \oplus \langle\mathcal{O}\rangle_K$.
c) For a polynomial $f = c_1u_1 + \cdots + c_su_s \neq 0$ with $c_i \in K \setminus \{0\}$ and $u_i \in \mathbb{T}^n$, the polynomial $\mathrm{BF}_{\mathcal{O}}(f) = \sum_{\{i \mid \mathrm{ind}_{\mathcal{O}}(u_i)\ \mathrm{max.}\}} c_iu_i$ is called the *border form* of f.
d) The ideal $\mathrm{BF}_{\mathcal{O}}(I) = (\mathrm{BF}_{\mathcal{O}}(f) \mid f \in I \setminus \{0\})$ is called the *border form ideal* of I.
e) The monomial ideal generated by $\partial\mathcal{O}$ is called the *border term ideal* of $\mathcal{O}$ and is denoted by $\mathrm{BT}_{\mathcal{O}}$.

Notice that if I has an $\mathcal{O}$-border basis, its border form ideal is $\mathrm{BF}_{\mathcal{O}}(I) = \mathrm{BT}_{\mathcal{O}}$. Thus our goal is to use a border basis of I to deform the ideal to its border form ideal. If the order ideal is of the form $\mathcal{O}_\sigma(I) = \mathbb{T}^n \setminus \mathrm{LT}_\sigma(I)$ for some term ordering σ, the Gröbner deformation can be used as follows.

Proposition 2.2. *Let σ be a term ordering, let $G = \{g_1, \dots, g_\nu\}$ be the $\mathcal{O}_\sigma(I)$-border basis of I, and let b_i the border term in the support of g_i for $i = 1, \dots, \nu$.*

a) *There exist weights* $W = (w_1, \dots, w_n) \in (\mathbb{N}_+)^n$ *such that* $b_j = \mathrm{DF}_W(g_j)$ *and* G *is a Macaulay basis of* I *with respect to the grading given by* W*.*
b) *Let* $\overline{P} = K[x_0, \dots, x_n]$ *be graded by* $\overline{W} = (1, w_1, \dots, w_n)$*. Then the ring* $\overline{P}/I^{\mathrm{hom}} = \overline{P}/(g_1^{\mathrm{hom}}, \dots, g_\nu^{\mathrm{hom}})$ *is a graded free* $K[x_0]$*-module.*

In particular, we have a flat family $K[x_0] \longrightarrow \overline{P}/I^{\mathrm{hom}}$ *whose* general fiber *is isomorphic to* $P/I \cong \overline{P}/(I^{\mathrm{hom}} + (x_0 - 1))$*, where* $I = (g_1, \dots, g_\nu)$*, and whose* special fiber *is isomorphic to* $P/\operatorname{BT}_{\mathcal{O}_\sigma(I)} \cong \overline{P}/(I^{\mathrm{hom}} + (x_0))$*.*

Proof. The first claim in a) follows from [5], Prop. 15.16. The second claim in a) is then a consequence of [17], Props. 6.4.18 and 4.2.15. The remaining claims follow from a) and [17], Thm. 4.3.22 and Prop. 4.3.23. □

For more general order ideals $\mathcal{O}$, i.e. for order ideals which are not necessarily of the form $\mathcal{O}_\sigma(I)$, one strategy is to deform a given $\mathcal{O}$-border basis of I first to a border basis of the degree form ideal $\mathrm{DF}_W(I)$ of I with respect to a suitably chosen grading. A border basis of $\mathrm{DF}_W(I)$ is always homogeneous, as the following lemma shows.

Lemma 2.3. *Let* P *be graded by a matrix* $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$*, let* $\mathcal{O}$ *be an order ideal, and let* $I \subset P$ *be a homogeneous ideal which has an* $\mathcal{O}$*-border basis. Then this* $\mathcal{O}$*-border basis of* I *consists of homogeneous polynomials.*

Proof. Let $\mathcal{O} = \{t_1, \dots, t_\mu\}$, let $b_j \in \partial\mathcal{O}$, and let $g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$ be the corresponding border basis element, where $c_{ij} \in K$. If we restrict the sum to those indices i for which $\deg_W(t_i) = \deg_W(b_j)$, we obtain a homogeneous element of I of the form $\tilde{g}_i = b_j - \sum_k c_{ik} t_k$. Now the uniqueness of the $\mathcal{O}$-border basis of I (cf. [17], 6.4.17) implies $g_i = \tilde{g}_i$. □

As for our idea to deform a border basis of I to a homogeneous border basis of $\mathrm{DF}_W(I)$, we have the following result.

Theorem 2.4. (Deformation to the Degree Form Ideal)
Let $W = (w_1, \dots, w_n) \in \mathrm{Mat}_{1,n}(\mathbb{N}_+)$ *be a row of positive integers, let* P *be graded by* W*, and let* $I \subset P$ *be a zero-dimensional ideal. Then the following conditions are equivalent.*

a) *The ideal* I *has an* $\mathcal{O}$*-border basis, say* $G = \{g_1, \dots, g_\nu\}$*, and we have* $b_j \in \mathrm{Supp}(\mathrm{DF}_W(g_j))$ *for* $j = 1, \dots, \nu$*.*
b) *The degree form ideal* $\mathrm{DF}_W(I)$ *has an* $\mathcal{O}$*-border basis.*

If these conditions are satisfied, the $\mathcal{O}$*-border basis of* $\mathrm{DF}_W(I)$ *is* $\mathrm{DF}_W(G) = \{\mathrm{DF}_W(g_1), \dots, \mathrm{DF}_W(g_s)\}$ *and there is a flat family* $K[x_0] \longrightarrow \overline{P}/I^{\mathrm{hom}}$ *whose general fiber is isomorphic to* P/I*, where* $I = (g_1, \dots, g_\nu)$*, and whose special fiber is isomorphic to* $P/\mathrm{DF}_W(I)$*, where* $\mathrm{DF}_W(I) = (\mathrm{DF}_W(g_1), \dots, \mathrm{DF}_W(g_\nu))$*.*

Proof. First we show that a) implies b). Since G is an $\mathcal{O}$-border basis of I and since $b_j \in \mathrm{Supp}(\mathrm{DF}_W(g_j))$ for $j = 1, \dots, \nu$, the set $\mathrm{DF}_W(G) = \{\mathrm{DF}_W(g_1), \dots, \mathrm{DF}_W(g_\nu)\}$ is an $\mathcal{O}$-border prebasis of the ideal $J = (\mathrm{DF}_W(g_1), \dots, \mathrm{DF}_W(g_\nu))$. By the Border Division Algorithm (see [17], Prop. 6.4.11), the residue classes of the elements of $\mathcal{O}$ generate the K-vector space P/J. Together with $J \subseteq \mathrm{DF}_W(I)$, this shows

$$\#\mathcal{O} = \dim_K(P/I) = \dim_K(P/\mathrm{DF}_W(I)) \le \dim_K(P/J) \le \#\mathcal{O}.$$

Therefore we get $J = \mathrm{DF}_W(I)$ and the residue classes of the elements of $\mathcal{O}$ are a K-basis of $P/\mathrm{DF}_W(I)$. From this the claim follows immediately.

Now we prove that b) implies a). Let σ be a term ordering on $\mathbb{T}^n$ which is compatible with the grading defined by W, and let $H = \{h_1, \dots, h_\nu\}$ be the $\mathcal{O}_\sigma(I)$-border basis of I. For the purposes of this proof, we may consider $\mathcal{O}$ and $\mathcal{O}_\sigma(I)$ as deg-ordered tuples (see [17], 4.5.4).

The fact that H is a σ-Gröbner basis of I implies by [17], 4.2.15 that H is a Macaulay basis of I with respect to the grading given by W. Then [17], 4.3.19 shows that I^{hom} is generated by $\{h_1^{\mathrm{hom}}, \dots, h_\nu^{\mathrm{hom}}\}$, and by [17], 4.3.22 the ring $\overline{P}/I^{\mathrm{hom}}$ is a graded free $K[x_0]$-module, where $K[x_0]$ is graded by $\deg(x_0) = 1$ and $\overline{P} = K[x_0, \dots, x_n]$ is graded by $\overline{W} = (1, w_1, \dots, w_n)$. More precisely, the proof of [17], 4.3.22 shows that the residue classes $\overline{\mathcal{O}_\sigma(I)}$ form a homogeneous $K[x_0]$-basis of this graded free module. Since the residue classes $\overline{\mathcal{O}}$ are homogeneous elements in $\overline{P}/I^{\mathrm{hom}}$, we can write $\overline{\mathcal{O}} = \overline{\mathcal{O}_\sigma(I)} \cdot \mathcal{A}$ with a homogeneous matrix $\mathcal{A} \in \mathrm{Mat}_\nu(K[x_0])$ (see [17], 4.7.1 and 4.7.3).

By the hypothesis, $\mathrm{DF}_W(I)$ has an $\mathcal{O}$-border basis. Thus the residue classes of the elements of $\mathcal{O}$ are a homogeneous K-basis of $P/\mathrm{DF}_W(I)$. Since also the residue classes of the elements of $\mathcal{O}_\sigma(I)$ are a homogeneous K-basis of this ring, the degree tuples of $\mathcal{O}$ and of $\mathcal{O}_\sigma(I)$ are identical. Therefore the matrix $\mathcal{A}$ is a block matrix of the form

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1q} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{A}_{q-1\,q} \\ 0 & \cdots & 0 & \mathcal{A}_{qq} \end{pmatrix}$$

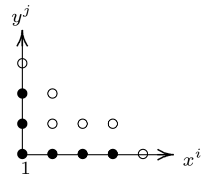
with square matrices $\mathcal{A}_{ii}$ having constant entries. Hence we have $\det(\mathcal{A}) \in K$, and the fact that the transformation matrix $\mathcal{A}|_{x_0 \mapsto 0}$ between the two homogeneous bases of $P/\mathrm{DF}_W(I)$ is invertible implies $\det(\mathcal{A}) \neq 0$. Altogether, it follows that $\overline{\mathcal{O}}$ is a homogeneous $K[x_0]$-basis of $\overline{P}/I^{\mathrm{hom}}$, too. In particular, the residue classes of $\mathcal{O}$ form a K-basis of $P/I \cong \overline{P}/(I^{\mathrm{hom}} + (x_0 - 1))$, i.e. the ideal I has an $\mathcal{O}$-border basis.

For every $j \in \{1, \dots, \nu\}$, we have a representation $b_j = \sum_{i=1}^{\mu} f_{ij} t_i + h_j$ with homogeneous polynomials $f_{ij} \in K[x_0]$ of degree $\deg_W(b_j) - \deg_W(t_i)$ and with a homogeneous polynomial $h_j \in I^{\mathrm{hom}}$ of degree $\deg_W(b_j)$. Setting $x_0 \mapsto 1$ in this representation, we find $g_j = b_j - \sum_{i=1}^{\mu} f_{ij}(1)\, t_i \in I$. It follows that these polynomials form the $\mathcal{O}$-border basis of I. By construction, we have $b_j \in \mathrm{Supp}(\mathrm{DF}_W(g_j))$.

The first additional claim is a consequence of the observation that $\mathrm{DF}_W(G)$ is an $\mathcal{O}$-border prebasis of $\mathrm{DF}_W(I)$ and of [17], Prop. 6.4.17. To construct the desired flat family, we use the fact that G is a Macaulay basis of I by what we have just shown and conclude from [17], 4.3.19 that $I^{\mathrm{hom}} = (g_1^{\mathrm{hom}}, \dots, g_\nu^{\mathrm{hom}})$. From this the claim follows. □

Let us look at an example for this proposition.

Example 2.5. Consider the ideal $I = (-2x^2 + xy - y^2 - 1,\, 8y^3 + 10x + 9y)$ in the polynomial ring $P = \mathbb{Q}[x, y]$. The degree form ideal of I with respect to the standard grading, i.e. the grading defined by $W = (1\ 1)$, is $\mathrm{DF}_W(I) = (-2x^2 + xy - y^2,\, y^3)$. We want to use the order ideal $\mathcal{O} = \{1, x, x^2, x^3, y, y^2\}$ whose border is given by $\partial\mathcal{O} = \{xy, y^3, xy^2, x^2y, x^3y, x^4\}$.



It is easy to check that $\mathrm{DF}_W(I)$ has an $\mathcal{O}$-border basis, namely $H = \{h_1, \dots, h_6\}$ with $h_1 = xy - 2x^2 - y^2$, $h_2 = y^3$, $h_3 = xy^2 + 4x^3$, $h_4 = x^2y + 2x^3$, $h_5 = x^3y$, and $h_6 = x^4$. Therefore the proposition says that I has an $\mathcal{O}$-border basis $G = \{g_1, \dots, g_6\}$, and that $h_i = \mathrm{DF}_W(g_i)$ for $i = 1, \dots, 6$. Indeed, if we compute this border basis we find that it is given by $g_1 = xy - 2x^2 - y^2 - 1$, $g_2 = y^3 + \frac{5}{4}x + \frac{9}{8}y$, $g_3 = xy^2 + 4x^3 + \frac{3}{4}x - \frac{1}{8}y$, $g_4 = x^2y + 2x^3 - \frac{1}{4}x - \frac{1}{8}y$, $g_5 = x^3y - \frac{1}{2}x^2 - \frac{1}{8}y^2 - \frac{3}{32}$, and $g_6 = x^4 - \frac{1}{64}$.

An easy modification of this example shows that the converse implication is not true without the hypothesis $b_j \in \mathrm{Supp}(\mathrm{DF}_W(g_j))$, i.e. that an $\mathcal{O}$-border basis of I does not necessarily deform to an $\mathcal{O}$-border basis of $\mathrm{DF}_W(I)$.

Example 2.6. Consider the ideal $I = (x^2y,\, x^3 - \frac{1}{2}xy,\, xy^2,\, y^3)$ in $P = \mathbb{Q}[x,y]$. With respect to the standard grading, we have $\mathrm{DF}_W(I) = (x^3, x^2y, xy^2, y^3)$. The ideal $\mathrm{DF}_W(I)$ does not have an $\mathcal{O}$-border basis for $\mathcal{O} = \{1, x, x^2, x^3, y, y^2\}$. However, the ideal I has the $\mathcal{O}$-border basis $G = \{g_1, \dots, g_6\}$, where $g_1 = xy - 2x^3$, $g_2 = y^3$, $g_3 = xy^2$, $g_4 = x^2y$, $g_5 = x^3y$, and $g_6 = x^4$.

The main reason why the last example exists is that one of the terms in $\mathcal{O}$ has a larger degree than the term xy in the border of $\mathcal{O}$. This suggests the following notion.

Definition 2.7. Let P be graded by a matrix $W \in \mathrm{Mat}_{1,n}(\mathbb{N}_+)$. The order ideal $\mathcal{O}$ is said to have a *maxdeg$_W$ border* if $\deg_W(b_j) \ge \deg_W(t_i)$ for $i = 1, \dots, \mu$ and $j = 1, \dots, \nu$. In other words, no term in $\mathcal{O}$ is allowed to have a degree larger than any term in the border.

Note that this condition is violated in Example 2.6. By choosing suitable weights, many order ideals can be seen to have a maxdeg_W border.

Example 2.8. Let $a \ge 1$, and let $\mathcal{O} = \{1, x_1, x_1^2, \dots, x_1^a\} \subset \mathbb{T}^n$. Then $\mathcal{O}$ has a maxdeg_W border with respect to the grading given by $W = (1\ a\ \cdots\ a)$.

One consequence of an order ideal having a maxdeg_W border is that $b_j \in \mathrm{Supp}(\mathrm{LF}_W(g_j))$ for $j = 1, \dots, \nu$ and every $\mathcal{O}$-border prebasis $G = \{g_1, \dots, g_\nu\}$. Thus the proposition applies in particular to order ideals having a maxdeg_W border. Let us end this section with an example for this part of the proposition.

Example 2.9. Let $\mathcal{O} = \{1, x, x^2, y, y^2\} \subset \mathbb{T}^2$. Then we have $\mathcal{O} = \mathbb{T}^2_{\le 2} \setminus \{xy\}$, i.e. the order ideal $\mathcal{O}$ has a maxdeg_W border with respect to the standard grading.

Consider the ideal $I = (x^2 + xy - \frac{1}{2}y^2 - x - \frac{1}{2}y,\, y^3 - y,\, xy^2 - xy)$ which is the vanishing ideal of the point set $\mathbb{X} = \{(0,0), (0,-1), (1,0), (1,1), (-1,1)\}$ if $\mathrm{char}(K) \ne 2$. We have $\partial\mathcal{O} = \{b_1, b_2, b_3, b_4, b_5\}$ with $b_1 = x^3$, $b_2 = x^2y$, $b_3 = xy$, $b_4 = xy^2$, and $b_5 = y^3$. The ideal I has an $\mathcal{O}$-border basis, namely $G = \{g_1, g_2, g_3, g_4, g_5\}$ with $g_1 = x^3 - x$, $g_2 = x^2y - \frac{1}{2}y^2 - \frac{1}{2}y$, $g_3 = xy + x^2 - \frac{1}{2}y^2 - x - \frac{1}{2}y$, $g_4 = xy^2 + x^2 - \frac{1}{2}y^2 - x - \frac{1}{2}y$, and $g_5 = y^3 - y$.

The order ideal $\mathcal{O}$ is not of the form $\mathcal{O} = \mathcal{O}_\sigma(I)$ for any term ordering σ. Using the proposition, we deform the border basis elements in G to their degree forms. Thus the ideal $\mathrm{DF}_W(I) = (x^3,\, x^2y,\, xy + x^2 - \frac{1}{2}y^2,\, xy^2,\, y^3)$ is a flat deformation of I and these five polynomials are an $\mathcal{O}$-border basis of $\mathrm{DF}_W(I)$. The task of deforming the homogeneous ideal $\mathrm{DF}_W(I)$ further to the border term ideal $\mathrm{BT}_{\mathcal{O}} = (x^3,\, x^2y,\, xy,\, xy^2,\, y^3)$ will be considered in Example 5.4.

3. The Border Basis Scheme

Let $\mathcal{O} = \{t_1, \dots, t_\mu\}$ be an order ideal in $\mathbb{T}^n$, and let $\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$ be its border. In this section we define a moduli space for *all* zero-dimensional ideals having an $\mathcal{O}$-border basis, and we use rational curves on this scheme to construct flat deformations of border bases.

Definition 3.1. Let $\{c_{ij} \mid 1 \le i \le \mu,\ 1 \le j \le \nu\}$ be a set of new indeterminates.

a) The *generic $\mathcal{O}$-border prebasis* is the set of polynomials $G = \{g_1, \dots, g_\nu\}$ in $K[x_1, \dots, x_n, c_{11}, \dots, c_{\mu\nu}]$ given by
$$g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$$
b) For $k = 1, \dots, n$, let $\mathcal{A}_k \in \mathrm{Mat}_\mu(K[c_{ij}])$ be the k^{th} formal multiplication matrix associated to G (cf. [17], Def. 6.4.29). It is also called the k^{th} *generic multiplication matrix* with respect to $\mathcal{O}$.
c) The affine scheme $\mathbb{B}_{\mathcal{O}} \subseteq \mathbb{A}^{\mu\nu}$ defined by the ideal $I(\mathbb{B}_{\mathcal{O}})$ generated by the entries of the matrices $\mathcal{A}_k\mathcal{A}_\ell - \mathcal{A}_\ell\mathcal{A}_k$ with $1 \le k < \ell \le n$ is called the *$\mathcal{O}$-border basis scheme*.
d) The coordinate ring $K[c_{11}, \dots, c_{\mu\nu}]/I(\mathbb{B}_{\mathcal{O}})$ of the scheme $\mathbb{B}_{\mathcal{O}}$ will be denoted by $B_{\mathcal{O}}$.

By [17], Thm. 6.4.30, a point $(\alpha_{ij}) \in K^{\mu\nu}$ yields a border basis $\sigma(G)$ when we apply the substitution $\sigma(c_{ij}) = \alpha_{ij}$ to G if and only if $\sigma(\mathcal{A}_k)\,\sigma(\mathcal{A}_\ell) = \sigma(\mathcal{A}_\ell)\,\sigma(\mathcal{A}_k)$ for $1 \le k < \ell \le n$. Therefore the K-rational points of $\mathbb{B}_{\mathcal{O}}$ are in 1–1 correspondence with the $\mathcal{O}$-border bases of zero-dimensional ideals in P, and thus with all zero-dimensional ideals having an $\mathcal{O}$-border basis.

Remark 3.2. (Properties of Border Basis Schemes)
Currently, not much seems to be known about border basis schemes. For instance, it is not clear which of them are connected, reduced or irreducible. Here we collect some basic observations.

a) By definition, the ideal $I(\mathbb{B}_{\mathcal{O}})$ is generated by polynomials of degree two.
b) The scheme $\mathbb{B}_{\mathcal{O}}$ can be embedded as an open affine subscheme of the Hilbert scheme parametrizing subschemes of $\mathbb{A}^n$ of length μ (see [19], Section 18.4).
c) There is an irreducible component of $\mathbb{B}_{\mathcal{O}}$ of dimension $n\mu$ which is the closure of the set of radical ideals having an $\mathcal{O}$-border basis.
d) The dimension of $\mathbb{B}_{\mathcal{O}}$ is claimed to be $n\mu$ in [22], Prop. 8.13. Example 5.6 shows that A. Iarrobino's example of a high-dimensional component of the Hilbert scheme yields a counterexample to this claim. It follows that the border basis scheme is in general not irreducible.
e) For every term ordering σ, there is a subset of $\mathbb{B}_{\mathcal{O}}$ which parametrizes all ideals I such that $\mathcal{O} = \mathcal{O}_\sigma(I)$. These subsets have turned out to be useful for

studying the Hilbert scheme parametrizing subschemes of $\mathbb{A}^n$ of length μ (see for instance [4] and [21]).

f) In the case $n = 2$ more precise information is available: for instance, it is known that $\mathbb{B}_{\mathcal{O}}$ is reduced, irreducible and smooth of dimension 2μ (see [7], [9] and [19], Ch. 18).

As usual, a moduli space such as the border basis scheme comes together with a universal family. In the present setting it is defined as follows.

Definition 3.3. Let $G = \{g_1, \dots, g_\nu\} \subset K[x_1, \dots, x_n, c_{11}, \dots, c_{\mu\nu}]$ with $g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$ for $j = 1, \dots, \nu$ be the generic $\mathcal{O}$-border prebasis. The ring $K[x_1, \dots, x_n, c_{11}, \dots, c_{\mu\nu}]/(I(\mathbb{B}_{\mathcal{O}}) + (g_1, \dots, g_\nu))$ will be denoted by $U_{\mathcal{O}}$. Then the natural homomorphism of K-algebras

$$\Phi : \; B_{\mathcal{O}} \longrightarrow U_{\mathcal{O}} \cong B_{\mathcal{O}}[x_1, \dots, x_n]/(g_1, \dots, g_\nu)$$

is called the *universal $\mathcal{O}$-border basis family.*

The fibers of the universal $\mathcal{O}$-border basis family are precisely the quotient rings P/I for which I is a zero-dimensional ideal which has an $\mathcal{O}$-border basis. The special fiber, i.e. the fiber corresponding to $(c_{11}, \dots, c_{\mu\nu})$, is the ring $P/\operatorname{BT}_{\mathcal{O}}$. It is the only fiber in the family which is defined by a monomial ideal. Although it is known that the universal family is free with basis $\mathcal{O}$ (see [6] or [10]), we believe that the following proof which generalizes the method in [20] is very elementary and conceptually simple.

Theorem 3.4. (The Universal Border Basis Family)
Let $\Phi : B_{\mathcal{O}} \longrightarrow U_{\mathcal{O}}$ be the universal $\mathcal{O}$-border basis family. Then the residue classes of the elements of $\mathcal{O}$ are a $B_{\mathcal{O}}$-module basis of $U_{\mathcal{O}}$. In particular, the map Φ is a flat homomorphism.

Proof. First we prove that the residue classes $\overline{\mathcal{O}}$ are a system of generators of the $B_{\mathcal{O}}$-module $U_{\mathcal{O}} \cong B_{\mathcal{O}}[x_1, \dots, x_n]/(G)$ where $G = \{g_1, \dots, g_\nu\}$ is the generic $\mathcal{O}$-border prebasis. In order to show that the map $\omega : B_{\mathcal{O}}^\nu \longrightarrow U_{\mathcal{O}}$ defined by $e_i \mapsto \bar{t}_i$ is surjective, we may extend the base field and hence assume that K is algebraically closed. By the local-global principle and the lemma of Nakayama, it suffices to show that the induced map

$$\bar{\omega} : \left((B_{\mathcal{O}})_{\mathfrak{m}} / \mathfrak{m} (B_{\mathcal{O}})_{\mathfrak{m}} \right)^\nu \longrightarrow (B_{\mathcal{O}})_{\mathfrak{m}}[x_1, \dots, x_n]/((G) + \mathfrak{m}(B_{\mathcal{O}})_{\mathfrak{m}}[x_1, \dots, x_n])$$

is surjective for every maximal ideal $\mathfrak{m} = (c_{ij} - \alpha_{ij})_{i,j}$ in $B_{\mathcal{O}}$. In other words, we need to show that the map ω becomes surjective if we substitute values $\alpha_{ij} \in K$ for the indeterminates c_{ij} and if these values have the property that the maximal ideal $(c_{ij} - \alpha_{ij})_{i,j}$ contains $I(\mathbb{B}_{\mathcal{O}})$. Thus the claim follows from the fact that G becomes an $\mathcal{O}$-border basis after such a substitution, since its associated formal multiplication matrices commute.

Now we show that $\overline{\mathcal{O}}$ is $B_{\mathcal{O}}$-linearly independent. We consider the free $B_{\mathcal{O}}$-submodule $M = \bigoplus_{i=1}^{\mu} B_{\mathcal{O}}\, t_i$ of $B_{\mathcal{O}}[x_1, \dots, x_n]$ and proceed in the following manner.

(1) We equip M with a suitable $B_{\mathcal{O}}[x_1, \dots, x_n]$-module structure.
(2) We show that this $B_{\mathcal{O}}[x_1, \dots, x_n]$-module is cyclic and construct a surjective $B_{\mathcal{O}}[x_1, \dots, x_n]$-linear map $\Theta : B_{\mathcal{O}}[x_1, \dots, x_n] \longrightarrow M$ which maps t_i to t_i.
(3) We prove that the kernel of Θ is precisely (G).

Altogether, it follows that Θ induces a map $\overline{\Theta} : B_{\mathcal{O}}[x_1, \ldots, x_n]/(G) \longrightarrow M$ which is an isomorphism of $B_{\mathcal{O}}$-modules and maps $\bar{t}_i$ to t_i. Thus $\overline{\mathcal{O}} = \{\bar{t}_1, \ldots, \bar{t}_\mu\}$ is a $B_{\mathcal{O}}$-basis of $U_{\mathcal{O}}$, as claimed.

To do Step 1, we let $\overline{\mathcal{A}}_j$ be the image of the generic multiplication matrix in $\mathrm{Mat}_\mu(B_{\mathcal{O}})$. Then we define

$$a * \sum_{i=1}^{\mu} a_i t_i = (t_1, \ldots, t_\mu) \cdot a \overline{\mathcal{I}}_\mu \cdot (a_1, \ldots, a_\mu)^{\mathrm{tr}} = \sum_{i=1}^{\mu} a\, a_i\, t_i \tag{1}$$

$$x_j * \sum_{i=1}^{\mu} a_i t_i = (t_1, \ldots, t_\mu) \cdot \overline{\mathcal{A}}_j \cdot (a_1, \ldots, a_\mu)^{\mathrm{tr}} \tag{2}$$

for $a, a_1, \ldots, a_\mu \in B_{\mathcal{O}}$ and $j = 1, \ldots, n$. Using this definition, the equalities

$$x_k x_j * \sum_{i=1}^{\mu} a_i t_i = x_k * (x_j * \sum_{i=1}^{\mu} a_i t_i) = (t_1, \ldots, t_\mu) \cdot \overline{\mathcal{A}}_k \overline{\mathcal{A}}_j \cdot (a_1, \ldots, a_\mu)^{\mathrm{tr}} \tag{3}$$

and the fact that the matrices $\overline{\mathcal{A}}_j$ commute show that this definition equips M with the structure of a $B_{\mathcal{O}}[x_1, \ldots, x_n]$-module. By using induction, we get

$$f * \sum_{i=1}^{\mu} a_i t_i = (t_1, \ldots, t_\mu) \cdot f(\overline{\mathcal{A}}_1, \ldots, \overline{\mathcal{A}}_n) \cdot (a_1, \ldots, a_\mu)^{\mathrm{tr}} \tag{4}$$

for every $f \in B_{\mathcal{O}}[x_1, \ldots, x_n]$ and all $a_1, \ldots, a_\mu \in B_{\mathcal{O}}$.

For Step 2, we assume w.l.o.g. that $t_1 = 1$. Using induction on $\deg(t_i)$, we want to show that $t_i * t_1 = t_i$ for $i = 1, \ldots, \mu$. The case $t_i = 1$ follows from (1). For the induction step, we write $t_i = x_k t_\ell$ and using (2), (3) and (4) we calculate

$$t_i * t_1 = x_k * (t_\ell * t_1) = x_k * t_\ell = (t_1, \ldots, t_\mu) \cdot \overline{\mathcal{A}}_k \cdot e_\ell^{\mathrm{tr}} = (t_1, \ldots, t_\mu) \cdot e_i^{\mathrm{tr}} = t_i$$

It follows that M is a cyclic $B_{\mathcal{O}}[x_1, \ldots, x_n]$-module generated by t_1. Thus we obtain a surjective $B_{\mathcal{O}}[x_1, \ldots, x_n]$-linear map $\Theta : B_{\mathcal{O}}[x_1, \ldots, x_n] \longrightarrow M$ which is defined by $f \mapsto f * t_1$. We have just shown that Θ satisfies $\Theta(t_i) = t_i$ for $i = 1, \ldots, \mu$.

Finally, to prove Step 3, we want to show that $\Theta(g_j) = 0$ for $j = 1, \ldots, \nu$. We write $b_j = x_k t_\ell$ and calculate $\Theta(g_i) = g_1 * t_1 = (t_1, \ldots, t_\mu) \cdot g_j(\overline{\mathcal{A}}_1, \ldots, \overline{\mathcal{A}}_n) \cdot e_1^{\mathrm{tr}}$. In particular, we get

$$\begin{aligned} g_j(\overline{\mathcal{A}}_1, \ldots, \overline{\mathcal{A}}_n) \cdot e_1^{\mathrm{tr}} &= b_j(\overline{\mathcal{A}}_1, \ldots, \overline{\mathcal{A}}_n) \cdot e_1^{\mathrm{tr}} - \sum_{i=1}^{\mu} c_{ij}\, t_i(\overline{\mathcal{A}}_1, \ldots, \overline{\mathcal{A}}_n) \cdot e_1^{\mathrm{tr}} \\ &= \overline{\mathcal{A}}_k \cdot t_\ell(\overline{\mathcal{A}}_1, \ldots, \overline{\mathcal{A}}_n) \cdot e_1^{\mathrm{tr}} - \sum_{i=1}^{\mu} c_{ij}\, e_i^{\mathrm{tr}} = \overline{\mathcal{A}}_k \cdot e_\ell^{\mathrm{tr}} - \sum_{i=1}^{\mu} c_{ij}\, e_i^{\mathrm{tr}} \\ &= \sum_{i=1}^{\mu} c_{ij}\, e_i^{\mathrm{tr}} - \sum_{i=1}^{\mu} c_{ij}\, e_i^{\mathrm{tr}} = 0 \end{aligned}$$

We have checked that $\Theta(g_j) = 0$ for $j = 1, \ldots, \nu$. Consequently, the map Θ induces a $B_{\mathcal{O}}$-linear map $\overline{\Theta} : B_{\mathcal{O}}[x_1, \ldots, x_n]/(G) \longrightarrow M$. We know already that $\overline{\mathcal{O}}$ generates the left-hand side and $\mathcal{O}$ is a $B_{\mathcal{O}}$-basis of the right-hand side. Hence the surjective map $\overline{\Theta}$ is also injective. □

In the remainder of this section we develop the connection between flat deformations over $K[z]$ of border bases and rational curves on the border basis scheme.

A rational curve on the $\mathcal{O}$-border basis scheme corresponds to a K-algebra homomorphism $\Psi : B_{\mathcal{O}} \longrightarrow K[z]$ of the corresponding affine coordinate rings. If we restrict the universal family of $\mathcal{O}$-border bases to this rational curve, we obtain the following flat deformation of border bases.

Corollary 3.5. *Let z be a new indeterminate, and let $\Psi : B_{\mathcal{O}} \longrightarrow K[z]$ be a homomorphism of K-algebras. By applying the base change Ψ to the universal family Φ, we get a homomorphism of $K[z]$-algebras*

$$\Phi_{K[z]} = \Phi \otimes_{B_{\mathcal{O}}} K[z] :\ K[z] \longrightarrow U_{\mathcal{O}} \otimes_{B_{\mathcal{O}}} K[z]$$

Then the residue classes of the elements of $\mathcal{O}$ form a $K[z]$-module basis of the right-hand side. In particular, the map $\Phi_{K[z]}$ defines a flat family.

This corollary can be used to construct flat deformations over $K[z]$ of border bases. Suppose the maximal ideal $\Psi^{-1}(z-1)$ corresponds to a given $\mathcal{O}$-border basis and the maximal ideal $\Psi^{-1}(z)$ is the ideal $(c_{11}, \dots, c_{\mu\nu})$ which corresponds to the border term ideal $(b_1, \dots, b_\nu)$. In other words, suppose that the rational curve connects a given point to the point $(0, \dots, 0)$ which corresponds to the border term ideal. Then the map $\Phi_{K[z]}$ defines a flat family over $K[z]$ whose generic fiber P/I is defined by the ideal I generated by the given $\mathcal{O}$-border basis and whose special fiber $P/(b_1, \dots, b_\nu)$ is defined by the border term ideal.

Another application of the theorem is the following criterion for checking the flatness of a family of border bases.

Corollary 3.6. (Flatness Criterion for Families of Border Bases)
Let z be a new indeterminate, let $\widetilde{P} = K[z][x_1, \dots, x_n]$, and let $g_j = b_j - \sum_{i=1}^{\mu} a_{ij}(z) t_i \in \widetilde{P}$ be polynomials with coefficients $a_{ij}(z) \in K[z]$. Let $\widetilde{I}$ be the ideal in $\widetilde{P}$ generated by $G = \{g_1, \dots, g_\nu\}$ and assume that the formal multiplication matrices $\mathcal{A}_k \in \mathrm{Mat}_\mu(K[z])$ of G are pairwise commuting.

- *a) For every $c \in K$, the set $\{g_1|_{z\mapsto c}, \dots, g_\nu|_{z\mapsto c}\}$ is an $\mathcal{O}$-border basis of the ideal $I_c = \widetilde{I}|_{z\mapsto c}$.*
- *b) The canonical K-algebra homomorphism*

$$\phi : \quad K[z] \ \longrightarrow\ K[z][x_1, \dots, x_n]/\widetilde{I}$$

defines a flat family. More precisely, the residue classes of the elements of $\mathcal{O}$ are a $K[z]$-basis of $K[z][x_1, \dots, x_n]/\widetilde{I}$.

Proof. First we show a). For every $c \in K$, the matrices $\mathcal{A}_k|_{z\mapsto c}$ are the multiplication matrices of $G|_{z\mapsto c}$. Thus the claim follows from [17], 6.4.30. Next we prove b). Since the matrices $\mathcal{A}_k$ commute, the map $B_{\mathcal{O}} \longrightarrow K[z]$ defined by $c_{ij} \mapsto a_{ij}(z)$ is a well-defined homomorphism of K-algebras. Hence it suffices to apply the preceding corollary. □

Remark 3.7. If K is infinite, the hypothesis that the formal multiplication matrices $\mathcal{A}_k$ commute can be replaced by the assumption that the matrices $\mathcal{A}_k|_{z\mapsto c}$ commute for every $c \in K$. This follows from the fact that a polynomial $f \in K[z]$ is zero if and only if $f(c) = 0$ for all $c \in K$.

Let us have a look at one particular border basis scheme in detail.

Example 3.8. Consider the case $n = 2$ and $\mathcal{O} = \{1, x, y, xy\}$. The border of $\mathcal{O}$ is $\partial\mathcal{O} = \{y^2, x^2, xy^2, x^2y\}$, so that in our terminology we have $\mu = 4$, $\nu = 4$, $t_1 = 1$, $t_2 = x$, $t_3 = y$, $t_4 = xy$, $b_1 = y^2$, $b_2 = x^2$, $b_3 = xy^2$, and $b_4 = x^2y$.

The generic multiplication matrices are

$$\mathcal{A}_x = \begin{pmatrix} 0 & c_{12} & 0 & c_{14} \\ 1 & c_{22} & 0 & c_{24} \\ 0 & c_{32} & 0 & c_{34} \\ 0 & c_{42} & 1 & c_{44} \end{pmatrix} \quad \text{and} \quad \mathcal{A}_y = \begin{pmatrix} 0 & 0 & c_{11} & c_{13} \\ 0 & 0 & c_{21} & c_{23} \\ 1 & 0 & c_{31} & c_{33} \\ 0 & 1 & c_{41} & c_{43} \end{pmatrix}$$

When we compute the ideal generated by the entries of $\mathcal{A}_x\mathcal{A}_y - \mathcal{A}_y\mathcal{A}_x$ and simplify its system of generators, we see that the ideal $I(\mathbb{B}_{\mathcal{O}})$ is generated by

$$\begin{aligned} \{&c_{23}c_{41}c_{42} - c_{21}c_{42}c_{43} + c_{21}c_{44} + c_{11} - c_{23}, \ \ -c_{21}c_{32} - c_{34}c_{41} + c_{33}, \\ &c_{34}c_{41}c_{42} - c_{32}c_{41}c_{44} + c_{32}c_{43} + c_{12} - c_{34}, \ \ -c_{21}c_{32} - c_{23}c_{42} + c_{24}, \\ &-c_{23}c_{32}c_{41} + c_{21}c_{32}c_{43} - c_{21}c_{34} + c_{13}, \ \ c_{21}c_{42} + c_{41}c_{44} + c_{31} - c_{43}, \\ &-c_{21}c_{34}c_{42} + c_{21}c_{32}c_{44} - c_{23}c_{32} + c_{14}, \ \ c_{32}c_{41} + c_{42}c_{43} + c_{22} - c_{44}\} \end{aligned}$$

Thus there are eight free indeterminates, namely c_{21}, c_{23}, c_{32}, c_{34}, c_{41}, c_{42}, c_{43}, and c_{44}, while the remaining indeterminates depend on the free ones by the polynomial expressions above. From this we conclude that the border basis scheme $\mathbb{B}_{\mathcal{O}}$ is an *affine cell* of the corresponding Hilbert scheme, i.e. an open subset which is isomorphic to an affine space. (This result is in agreement with [9], Cor. 7.3.2, but not with [19], Example 18.6.)

Its coordinate ring is explicitly represented by the isomorphism

$$B_{\mathcal{O}} \xrightarrow{\sim} K[c_{21}, c_{23}, c_{32}, c_{34}, c_{41}, c_{42}, c_{43}, c_{44}]$$

given by

$$\begin{aligned} c_{11} &\longmapsto -c_{23}c_{41}c_{42} + c_{21}c_{42}c_{43} - c_{21}c_{44} + c_{23} \\ c_{12} &\longmapsto -c_{34}c_{41}c_{42} + c_{32}c_{41}c_{44} - c_{32}c_{43} + c_{34} \\ c_{13} &\longmapsto c_{23}c_{32}c_{41} - c_{21}c_{32}c_{43} + c_{21}c_{34} \\ c_{14} &\longmapsto c_{21}c_{34}c_{42} - c_{21}c_{32}c_{44} + c_{23}c_{32} \\ c_{22} &\longmapsto -c_{32}c_{41} - c_{42}c_{43} + c_{44} \\ c_{24} &\longmapsto c_{21}c_{32} + c_{23}c_{42} \\ c_{31} &\longmapsto -c_{21}c_{42} - c_{41}c_{44} + c_{43} \\ c_{33} &\longmapsto c_{21}c_{32} + c_{34}c_{41} \end{aligned}$$

Hence we have $U_{\mathcal{O}} \cong K[x, y, c_{21}, c_{23}, c_{32}, c_{34}, c_{41}, c_{42}, c_{43}, c_{44}]/(\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3, \widetilde{g}_4)$ where

$$\begin{aligned} \widetilde{g}_1 &= y^2 - (-c_{23}c_{41}c_{42} + c_{21}c_{42}c_{43} - c_{21}c_{44} + c_{23}) \\ &\quad -c_{21}x - (-c_{21}c_{42} - c_{41}c_{44} + c_{43})y - c_{41}xy, \\ \widetilde{g}_2 &= x^2 - (-c_{34}c_{41}c_{42} + c_{32}c_{41}c_{44} - c_{32}c_{43} + c_{34}) \\ &\quad -(-c_{32}c_{41} - c_{42}c_{43} + c_{44})x - c_{32}y - c_{42}xy, \\ \widetilde{g}_3 &= xy^2 - (c_{23}c_{32}c_{41} - c_{21}c_{32}c_{43} + c_{21}c_{34}) \\ &\quad -c_{23}x - (c_{21}c_{32} + c_{34}c_{41})y - c_{43}xy, \\ \widetilde{g}_4 &= x^2y - (c_{21}c_{34}c_{42} - c_{21}c_{32}c_{44} + c_{23}c_{32}) \\ &\quad -(c_{21}c_{32} + c_{23}c_{42})x - c_{34}y - c_{44}xy, \end{aligned}$$

The ideal $(\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3, \widetilde{g}_4)$ is the defining ideal of the family of all subschemes of length four of the affine plane which have the property that their coordinate ring admits $\overline{\mathcal{O}}$ as a vector space basis. Since the border basis scheme is isomorphic to an affine space in this case, we can connect every point to the point corresponding to (x^2, y^2) by a rational curve. Therefore every ideal in the family can be deformed by a flat deformation to the monomial ideal (x^2, y^2). Algebraically, it suffices to substitute each free indeterminate c_{ij} with zc_{ij} where z is a new indeterminate. We get the K-algebra homomorphism

$$\Phi_{K[z]} : K[z] \longrightarrow K[x, y, z, c_{21}, c_{23}, c_{32}, c_{34}, c_{41}, c_{42}, c_{43}, c_{44}]/(\overline{g}_1, \overline{g}_2, \overline{g}_3, \overline{g}_4)$$

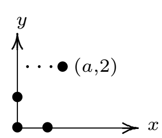
where

$$\begin{aligned}
\overline{g}_1 &= y^2 - (-z^3 c_{23} c_{41} c_{42} + z^3 c_{21} c_{42} c_{43} - z^2 c_{21} c_{44} + z c_{23}) \\
&\quad -z c_{21} x - (-z^2 c_{21} c_{42} - z^2 c_{41} c_{44} + z c_{43}) y - z c_{41} xy, \\
\overline{g}_2 &= x^2 - (-z^3 c_{34} c_{41} c_{42} + z^3 c_{32} c_{41} c_{44} - z^2 c_{32} c_{43} + z c_{34}) \\
&\quad -(-z^2 c_{32} c_{41} - z^2 c_{42} c_{43} + z c_{44}) x - z c_{32} y - z c_{42} xy, \\
\overline{g}_3 &= xy^2 - (z^3 c_{23} c_{32} c_{41} - z^3 c_{21} c_{32} c_{43} + z^2 c_{21} c_{34}) \\
&\quad -z c_{23} x - (z^2 c_{21} c_{32} + z^2 c_{34} c_{41}) y - z c_{43} xy, \\
\overline{g}_4 &= x^2 y - (z^3 c_{21} c_{34} c_{42} - z^3 c_{21} c_{32} c_{44} + z^2 c_{23} c_{32}) \\
&\quad -(z^2 c_{21} c_{32} + z^2 c_{23} c_{42}) x - z c_{34} y - z c_{44} xy,
\end{aligned}$$

By Corollary 3.5, this homomorphism is flat. For every point on the border basis scheme, it connects the corresponding ideal to $\mathrm{BT}_{\mathcal{O}} = (y^2, x^2, xy^2, x^2y) = (x^2, y^2)$.

The next example shows that natural families of ideals can lead us out of the affine open subset $\mathbb{B}_{\mathcal{O}}$ of the Hilbert scheme.

Example 3.9. Using $K = \mathbb{R}$ and $P = \mathbb{R}[x, y]$, we consider the family of reduced zero-dimensional schemes $\mathbb{X}_a = \{(a, 2),\ (0, 1),\ (0, 0),\ (1, 0)\} \subset \mathbb{R}^2$ with $a \in \mathbb{R}$.



For $\sigma = \texttt{DegRevLex}$, the reduced σ-Gröbner basis of the vanishing ideal $I_a \subset P$ of $\mathbb{X}_a$ is

$$G'_a = \{x^2 + \tfrac{1}{2}\, a(1-a) y^2 - x - \tfrac{1}{2}\, a(1-a)\, y,\ xy - ay^2 + ay,\ y^3 - 3y^2 + 2y\}$$

and thus we have $\mathcal{O}_\sigma(I_a) = \{1, x, y, y^2\}$. We may extend G'_a to an $\mathcal{O}_\sigma(I_a)$-border basis of I_a and get

$$G_a = G'_a \,\cup\, \{xy^2 - 2ay^2 + 2ay\}$$

The residue classes of the elements of $\mathcal{O}_\sigma(I_a)$ are a vector space basis of P/I_a for every $a \in \mathbb{R}$. We let $I = (x^2 + \frac{1}{2}\, z(1-z)y^2 - x - \frac{1}{2}\, z(1-z)y,\ xy - zy^2 + zy,\ y^3 - 3y^2 + 2y,\ xy^2 - 2zy^2 + 2zy) \subset P[z]$. Then the natural map $\mathbb{R}[z] \longrightarrow P[z]/I$ is a flat homomorphism whose fibers are the rings P/I_a. Thus the point corresponding to G_a on the border basis scheme $\mathbb{B}_{\mathcal{O}_\sigma(I_a)}$ is connected to the point representing G_0 via a rational curve.

Now we consider the order ideal $\mathcal{O} = \{1, x, y, xy\}$. For $a \neq 0$, the set $\mathbb{X}_a$ is a complete intersection of type $(2, 2)$. Its vanishing ideal I_a has an $\mathcal{O}$-border basis, namely

$$H_a = \{y^2 - \tfrac{1}{a}\,xy - y,\ xy^2 - 2xy,\ x^2y - axy,\ x^2 + \tfrac{1}{2}\,(1-a)xy - x\}$$

However, for $a = 0$, the ideal I_0 has no $\mathcal{O}$-border basis because $xy \in I_0$. One of the coefficients in H_a tends to ∞ as $a \longrightarrow 0$. This happens since the scheme $\mathbb{B}_\mathcal{O}$ is not complete.

4. Defining Equations for the Border Basis Scheme

The defining equations for the border basis scheme can be constructed in different ways. One construction is given by imposing the commutativity law to the multiplication matrices, as we have seen in the preceding section. Another construction was given in [10], and a different but related one in [13] and [22]. After describing this alternative construction, we use it to get rid of as many generators of the vanishing ideal of $\mathbb{B}_\mathcal{O}$ as possible and examine some claims in [22] in this regard.

Let $\mathcal{O} = \{t_1, \dots, t_\mu\}$ be an order ideal and $\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$ its border. In [13], Def. 17, two terms $b_i, b_j \in \partial\mathcal{O}$ are called *next-door neighbors* if $b_i = x_k b_j$ for some $k \in \{1, \dots, n\}$ and *across-the street neighbors* if $x_k b_i = x_\ell b_j$ for some $k, \ell \in \{1, \dots, n\}$. In addition to these notions we shall say that across-the-street neighbors b_i, b_j with $x_k b_i = x_\ell b_j$ are *across-the-corner neighbors* of there exists a term $b_m \in \partial\mathcal{O}$ such that $b_i = x_\ell b_m$ and $b_j = x_k b_m$.

In [22], Def. 8.5, the graph whose vertices are the border terms and whose edges are given by the neighbor relation is called the *border web* of $\mathcal{O}$. The Buchberger criterion for border bases (see [13], Prop. 18 and [22], Thm. 8.11) says that an $\mathcal{O}$-border prebasis $\{g_1, \dots, g_\nu\}$ with $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i$ and $a_{ij} \in K$ is an $\mathcal{O}$-border basis if and only if the S-polynomials $S(g_i, g_j)$ reduce to zero using G for all (i,j) such that b_i and b_j are neighbors. This characterization can be used to construct the equations defining the border basis scheme in an alternative way.

Proposition 4.1. (Lifting Neighbor Syzygies)
Let $G = \{g_1, \dots, g_\nu\}$ be the generic $\mathcal{O}$-border prebasis, where $g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i \in K[x_1, \dots, x_n, c_{11}, \dots, c_{\mu\nu}]$, let $\mathcal{A}_1, \dots, \mathcal{A}_n \in \mathrm{Mat}_\mu(K[c_{ij}])$ be the generic multiplication matrices with respect to $\mathcal{O}$, and let $c_j = (c_{1j}, \dots, c_{\mu j})^{\mathrm{tr}} \in \mathrm{Mat}_{\mu,1}(K[c_{ij}])$ for $j = 1, \dots, \nu$. Consider the following sets of polynomials in $K[c_{11}, \dots, c_{\mu\nu}]$:

1. *If $b_i, b_j \in \partial\mathcal{O}$ are next-door neighbors with $b_i = x_k b_j$, let $\mathrm{ND}(i,j)$ be the set of polynomial entries of $c_i - \mathcal{A}_k c_j$.*
2. *If $b_i, b_j \in \partial\mathcal{O}$ are across-the-street neighbors with $x_k b_i = x_\ell b_j$, let $\mathrm{AS}(i,j)$ be the set of polynomial entries of $\mathcal{A}_k c_i - \mathcal{A}_\ell c_j$.*

Then the following claims hold true.

a) The union of all sets $\mathrm{ND}(i,j)$ and all sets $\mathrm{AS}(i,j)$ contains the set of the nontrivial entries of the commutators $\mathcal{A}_k\mathcal{A}_\ell - \mathcal{A}_\ell\mathcal{A}_k$ with $1 \le k < \ell \le n$.

b) If one removes from this union all sets $\mathrm{AS}(i,j)$ such that b_i, b_j are across-the-corner neighbors, one gets precisely the set of the nontrivial entries of the commutators $\mathcal{A}_k\mathcal{A}_\ell - \mathcal{A}_\ell\mathcal{A}_k$ with $1 \le k < \ell \le n$. In particular, the remaining union generates the vanishing ideal $I(\mathbb{B}_\mathcal{O})$ of the $\mathcal{O}$-border basis scheme.

c) The polynomials in the sets $\mathrm{AS}(i,j)$ corresponding to across-the-corner neighbors b_i, b_j are contained in $I(\mathbb{B}_\mathcal{O})$.

Proof. First we prove a) and b). The S-polynomials $g_i - x_k g_j$ resp. $x_k g_i - x_\ell g_j$ are $K[c_{ij}]$-linear combinations of terms in $\mathcal{O} \cup \partial\mathcal{O}$. We want to find representations of these polynomials as $K[c_{ij}]$-linear combinations of elements of $\mathcal{O}$ only. Since we have $b_i - x_k b_j = 0$ resp. $x_k b_i - x_\ell b_j = 0$, we have to represent $(-\sum_{m=1}^{\mu} c_{mi} t_m) - x_k\,(-\sum_{m=1}^{\mu} c_{mj} t_m)$ resp. $x_k\,(-\sum_{m=1}^{\mu} c_{mi} t_m) - x_\ell\,(-\sum_{m=1}^{\mu} c_{mj} t_m)$ using $\mathcal{O}$. By the definition of the generic multiplication matrices, these representations are given by $(t_1, \dots, t_\mu) \cdot (c_i - \mathcal{A}_k c_j)$ resp. $(t_1, \dots, t_\mu) \cdot (\mathcal{A}_k c_i - \mathcal{A}_\ell c_j)$. The coefficients of the terms t_i in these representations are precisely the polynomials in $\mathrm{ND}(i,j)$ resp. in $\mathrm{AS}(i,j)$.

Now we consider the polynomials in the sets $\mathrm{ND}(i,j)$ and in the sets $\mathrm{AS}(i,j)$ for which b_i, b_j are not across-the-corner neighbors. The fact that these polynomials are exactly the nontrivial entries of the commutators $\mathcal{A}_k\mathcal{A}_\ell - \mathcal{A}_\ell\mathcal{A}_k$ was checked in [13], Section 4 resp. [22], Prop. 8.10.

It remains to show c). Let $b_i = x_\ell b_m$ and $b_j = x_k b_m$. By what we have shown so far, the polynomials which are the components of $c_i - \mathcal{A}_\ell c_m$ and $c_j - \mathcal{A}_k c_m$ are contained in $I(\mathbb{B}_\mathcal{O})$. Moreover, the polynomial entries of $\mathcal{A}_k\mathcal{A}_\ell - \mathcal{A}_\ell\mathcal{A}_k$ are in $I(\mathbb{B}_\mathcal{O})$. Therefore also the components of

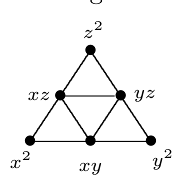
$$\mathcal{A}_k c_i - \mathcal{A}_\ell c_j = \mathcal{A}_k(c_i - \mathcal{A}_\ell c_m) + (\mathcal{A}_k\mathcal{A}_\ell - \mathcal{A}_\ell\mathcal{A}_k)c_m - \mathcal{A}_\ell(c_j - \mathcal{A}_k c_m)$$

are contained in $I(\mathbb{B}_\mathcal{O})$. These components are exactly the polynomials in $\mathrm{AS}(i,j)$. □

Another way of phrasing this proposition is to say that, for G to be a border basis, the neighbor syzygies $e_i - x_k e_j$ resp. $x_k e_i - x_\ell e_j$ of the border tuple $(b_1, \dots, b_\nu)$ have to lift to syzygies of $(g_1, \dots, g_\nu)$ and that the defining equations of $\mathcal{O}$ are precisely the equations expressing the existence of these liftings (see [13], Ex. 23). Now it is a well-known phenomenon in Gröbner basis theory that it suffices to lift a minimal set of generators of the syzygy module of the leading terms (see for instance [16], Prop. 2.3.10). In [22], Props. 8.14 and 8.15, an attempt was made to use a similar idea for removing unnecessary generators of $I(\mathbb{B}_\mathcal{O})$. However, the claims made there are not correct in general, as the following examples show.

The first example has surfaced in a number of different contexts; see the papers [12], [18] and the references therein.

Example 4.2. Let us consider $P = K[x,y,z]$ and $\mathcal{O} = \{1, x, y, z\}$. The border $\partial\mathcal{O} = \{b_1, \dots, b_6\}$ with $b_1 = x^2$, $b_2 = xy$, $b_3 = xz$, $b_4 = y^2$, $b_5 = yz$, and $b_6 = z^2$ has a very simple border web consisting of nine across-the-street neighbors:



These across-the-street neighbors yield $9 \cdot 4 = 36$ quadratic equations for $I(\mathbb{B}_\mathcal{O})$ in $K[c_{11}, \dots, c_{46}]$. Contrary to the claim in [22], Prop. 8.15, the equations for the neighbor pair (x^2, xy) are not contained in the ideal generated by the remaining 32 equations. In fact, in agreement with Proposition 4.5, it turns out that the four equations corresponding to the pair (xy, xz) are contained in the ideal generated by the eight equations corresponding to the two pairs (xy, yz) and (xz, yz) (see Example 4.7).

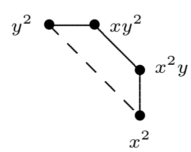
In order to see whether the ideal $I(\mathbb{B}_\mathcal{O})$ is a complete intersection (as claimed in [22], p. 297), we examine its generators more closely. If we define a grading by letting $\deg_W(c_{1j}) = 2$ for $j = 1, \ldots, 6$ and $\deg_W(c_{ij}) = 1$ for $i > 1$, the 36 generators are homogeneous with respect to the grading given by W. Every minimal system of generators of the ideal $I(\mathbb{B}_\mathcal{O})$ consists of 21 polynomials, while its height is 12. Hence it is very far from being a complete intersection.

The indeterminates $c_{11}, \ldots, c_{16}$ corresponding to the constant coefficients of the generic border basis form the linear parts of six of the 21 minimal generators and do not divide any of the other terms. We may eliminate them and obtain an ideal J in $Q = K[c_{21}, \ldots, c_{46}]$ which has (after interreduction) 15 homogeneous quadratic generators. Geometrically speaking, there is a projection to an 18-dimensional affine space which maps the border basis scheme isomorphically to a homogeneous subscheme of $\mathbb{A}^{18}$. In fact, it is known that this scheme is an affine cone with 3-dimensional vertex over the Grassmannian $\mathrm{Grass}(2,6) \subset \mathbb{P}^{14}$.

The ideal J is prime and the ring Q/J is Gorenstein with Hilbert series $(1 + 6z + 6z^2 + 1)/(1-z)^{12}$. The minimal number of generators of J is 15. The border basis scheme is irreducible and has the expected dimension, namely 12.

Also the lifting of trivial syzygies fails in the border basis scheme setting, as our next example shows (see also Example 3.8).

Example 4.3. Let $P = K[x,y]$ and $\mathcal{O} = \{1, x, y, xy\}$. Then the border of $\mathcal{O}$ is $\partial\mathcal{O} = \{x^2, y^2, x^2y, xy^2\}$. It has two next-door neighbors (x^2, x^2y), (y^2, xy^2) and one across-the-street neighbor (x^2y, xy^2). If one includes the "trivial syzygy pair" (x^2, y^2), there is one loop in the border web:

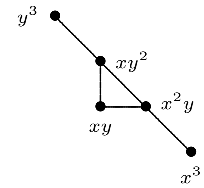


The neighbor pairs yield four equations each for the defining ideal of $\mathbb{B}_\mathcal{O}$. Contrary to a claim in [22], p. 297, one cannot drop one of these sets of four polynomials without changing the ideal. Thus the lifting of a "trivial" syzygy cannot be used to remove defining equations for the border basis scheme.

Interestingly, in the case at hand, the ideal $I(\mathbb{B}_\mathcal{O})$ is indeed a complete intersection: there exists a subset of 8 of the 12 equations which generates $I(\mathbb{B}_\mathcal{O})$ minimally and $\dim(K[c_{11}, \ldots, c_{44}]/I(\mathbb{B}_\mathcal{O})) = 8$. But the unnecessary generators are spread around the blocks coming from the neighbor pairs.

Our next example shows how one can sometimes get rid of some generators of $I(\mathbb{B}_\mathcal{O})$ using part c) of the proposition.

Example 4.4. Consider $P = K[x,y]$ and $\mathcal{O} = \{1, x, y, x^2, y^2\}$. Then we have $\partial\mathcal{O} = \{b_1, \ldots, b_5\}$ with $b_1 = y^3$, $b_2 = xy^2$, $b_3 = xy$, $b_4 = x^2y$, and $b_5 = x^3$, two next-door neighbors (xy, xy^2) and (xy, x^2y), two proper across-the-street neighbors (y^3, xy^2) and (x^2y, x^3), and one pair of across-the-corner neighbors (xy^2, x^2y). Thus the border web of $\mathcal{O}$ looks as follows.



Using part c) of the proposition, we know that $I(\mathbb{B}_\mathcal{O})$ is generated by $\mathrm{AS}(1,2)$, $\mathrm{AS}(4,5)$, $\mathrm{ND}(2,3)$, and $\mathrm{ND}(3,4)$. In fact, using CoCoA, we may check that none of these sets can be removed without changing the ideal.

On the positive side, the following proposition allows us to remove at least a few polynomials from the system of generators of $I(\mathbb{B}_\mathcal{O})$ given in Proposition 4.1.

Proposition 4.5. (Removing Redundant Generators of $I(\mathbb{B}_\mathcal{O})$)
Let $\mathcal{O} = \{t_1, \dots, t_\mu\}$ be an order ideal with border $\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$, and let H be a system of generators of $I(\mathbb{B}_\mathcal{O})$.

- *a) Suppose that there exist $i, j, k \in \{1, \dots, \nu\}$ and $\ell, m \in \{1, \dots, n\}$ such that $b_k = x_\ell b_i = x_m b_j$. If the sets $\mathrm{AS}(i,j)$, $\mathrm{ND}(i,k)$ and $\mathrm{ND}(j,k)$ are contained in H and one removes one of these sets, the remaining polynomials still generate $I(\mathbb{B}_\mathcal{O})$.*
- *b) Suppose that there exist $i, j, k \in \{1, \dots, \nu\}$ and $\alpha, \beta, \gamma \in \{1, \dots, n\}$ such that $x_\alpha b_i = x_\beta b_j = x_\gamma b_k$. If the sets $\mathrm{AS}(i,j)$, $\mathrm{AS}(i,k)$ and $\mathrm{AS}(j,k)$ are contained in H and one removes one of these sets, the remaining polynomials still generate $I(\mathbb{B}_\mathcal{O})$.*

Proof. Let $\mathcal{A}_1 \dots, \mathcal{A}_n$ be the generic multiplication matrices with respect to $\mathcal{O}$, and let $c_j = (c_{1j}, \dots, c_{\mu j})^{\mathrm{tr}} \in \mathrm{Mat}_{\mu,1}(K[c_{ij}])$ for $j = 1, \dots, \nu$.

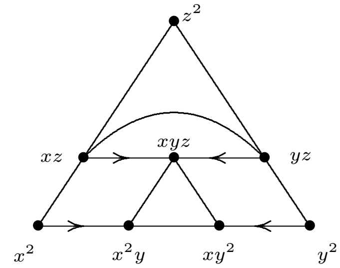
First we show a). The polynomials in $\mathrm{AS}(i,j)$ are the components of $\mathcal{A}_\ell \cdot c_i - \mathcal{A}_m \cdot c_j$, the polynomials in $\mathrm{ND}(i,k)$ are the components of $c_k - \mathcal{A}_\ell \cdot c_i$, and the polynomials in $\mathrm{ND}(j,k)$ are the components of $c_k - \mathcal{A}_m \cdot c_j$. Thus the claim follows from

$$(\mathcal{A}_\ell \cdot c_i - \mathcal{A}_m \cdot c_j) + (c_k - \mathcal{A}_\ell \cdot c_i) - (c_k - \mathcal{A}_m \cdot c_j) = 0$$

To show b), we argue similarly. The polynomials in $\mathrm{AS}(i,j)$ are the components of $\mathcal{A}_\alpha \cdot c_i - \mathcal{A}_\beta \cdot c_j$, the polynomials in $\mathrm{AS}(i,k)$ are the components of $\mathcal{A}_\alpha \cdot c_i - \mathcal{A}_\gamma \cdot c_k$, and the polynomials in $\mathrm{AS}(j,k)$ are the components of $\mathcal{A}_\beta \cdot c_j - \mathcal{A}_\gamma \cdot c_k$. □

Let us illustrate the application of this proposition with a couple of examples.

Example 4.6. Let $P = K[x,y,z]$ and $\mathcal{O} = [1, x, y, z, xy]$. Then we have $\partial\mathcal{O} = \{b_1, \dots, b_8\}$ with $b_1 = z^2$, $b_2 = yz$, $b_3 = xz$, $b_4 = y^2$, $b_5 = x^2$, $b_6 = xyz$, $b_7 = xy^2$, and $b_8 = x^2y$. There are four next-door neighbors (yz, xyz), (xz, xyz), $y^2, xy^2)$, (x^2, x^2y) and eight across-the-street neighbors (yz, z^2), (xz, z^2), (xz, yz), (y^2, yz), (x^2, xz), (xy^2, xyz), (x^2y, xyz), and (x^2y, xy^2). This yields the border web



where we have marked next-door neighbors by arrows. Since we have $xb_2 = yb_3 = b_6$, we can use part a) of the proposition and remove one of the sets $\mathrm{AS}(2,3)$, $\mathrm{ND}(2,6)$, or $\mathrm{ND}(3,6)$ from the system of generators of $I(\mathbb{B}_\mathcal{O})$. Although there are many further "loops" in the remaining part of the border web, we may use CoCoA to check that no other set $\mathrm{ND}(i,j)$ or $\mathrm{AS}(i,j)$ can be removed without changing the generated ideal.

Using the second part of the proposition, we can remove some generators of $I(\mathbb{B}_\mathcal{O})$ in Example 4.2.

Example 4.7. Consider $P = K[x,y,z]$ and $\mathcal{O} = \{1,x,y,z\}$ with the border web explained in Example 4.2. Then the border terms $b_2 = xy$, $b_3 = xz$ and $b_5 = yz$ satisfy $zb_2 = yb_3 = xb_5$. Therefore one of the sets $\mathrm{AS}(2,3)$, $\mathrm{AS}(2,5)$, or $\mathrm{AS}(3,5)$ can be removed from the system of generators of $I(\mathbb{B}_\mathcal{O})$ without changing the ideal. As already explained in Example 4.2, none of the remaining sets $\mathrm{AS}(i,j)$ can be removed thereafter.

5. The Homogeneous Border Basis Scheme

Let $P = K[x_1,\dots,x_n]$ be graded by $W = (w_1 \ \cdots \ w_n) \in \mathrm{Mat}_{1,n}(\mathbb{N}_+)$, let $\mathcal{O} = \{t_1,\dots,t_\mu\}$ be an order ideal, and let $\partial\mathcal{O} = \{b_1,\dots,b_\nu\}$ be its border. If we restrict our attention to zero-dimensional ideals $I \subset P$ which have an $\mathcal{O}$-border basis and are homogeneous with respect to the grading given by W, we obtain the following subscheme of the border basis scheme.

Definition 5.1. Let $\{c_{ij} \mid 1 \le i \le \mu,\ 1 \le j \le \nu\}$ be a set of new indeterminates.

a) The *generic homogeneous $\mathcal{O}$-border prebasis* is defined to be the set of polynomials $G = \{g_1,\dots,g_\nu\}$ in the ring $K[x_1,\dots,x_n,c_{11},\dots,c_{\mu\nu}]$ where
$$g_j = b_j - \sum_{\{i\in\{1,\dots,\mu\} \mid \deg_W(t_i)=\deg_W(b_j)\}} c_{ij}t_i$$
for $j = 1,\dots,\nu$.

b) For $k = 1,\dots,n$, let $\mathcal{A}_k \in \mathrm{Mat}_\mu(K[c_{ij}])$ be the k^{th} formal multiplication matrix associated to G. It is also called the k^{th} *generic homogeneous multiplication matrix* with respect to $\mathcal{O}$.

c) The affine scheme $\mathbb{B}_\mathcal{O}^{\mathrm{hom}} \subseteq \mathbb{A}^{\mu\nu}$, defined by the ideal $I(\mathbb{B}_\mathcal{O}^{\mathrm{hom}})$ generated by the entries of the matrices $\mathcal{A}_k\mathcal{A}_\ell - \mathcal{A}_\ell\mathcal{A}_k$ with $1 \le k < \ell \le n$ together with $\{c_{ij} \mid \deg_W(t_i) \ne \deg_W(b_j)\}$, is called the *homogeneous $\mathcal{O}$-border basis scheme*.

Clearly, the homogeneous border basis scheme is the intersection of $\mathbb{B}_\mathcal{O}$ with the linear space $\mathcal{Z}(c_{ij} \mid \deg_W(t_i) \ne \deg_W(b_j))$.

Remark 5.2. Let us equip $K[x_1,\dots,x_n,c_{11},\dots,c_{\mu\nu}]$ with the grading defined by the matrix $\overline{W}$ for which $\deg_{\overline{W}}(c_{ij})=0$ and $\deg_{\overline{W}}(x_i)=w_i$.

a) The matrix $\mathcal{A}_k$ is a homogeneous matrix in the sense of [17], Def. 4.7.1, with respect to the degree pair given by $(\deg_W(t_1),\dots,\deg_W(t_\mu))$ for the rows and $(\deg_W(x_kt_1),\dots,\deg_W(x_kt_\mu))$ for the columns.

b) As explained in [17], p. 118, we can add a vector $d\in\mathbb{Z}^\mu$ to a degree pair and still have a degree pair for the same homogeneous matrix. Thus the matrix $\mathcal{A}_\ell$ also has the degree pair given by $(\deg_W(x_kt_1),\dots,\deg_W(x_kt_\mu))$ for the rows and $(\deg_W(x_kx_\ell t_1),\dots,\deg_W(x_kx_\ell t_\mu))$ for the columns. In this way we see that both $\mathcal{A}_k\mathcal{A}_\ell$ and $\mathcal{A}_\ell\mathcal{A}_k$ are homogeneous matrices with respect to the degree pair given by $(\deg_W(t_1),\dots,\deg_W(t_\mu))$ for the rows and $(\deg_W(x_kx_\ell t_1),\dots,\deg_W(x_kx_\ell t_\mu))$ for the columns. Consequently, also the commutator $\mathcal{A}_k\mathcal{A}_\ell-\mathcal{A}_\ell\mathcal{A}_k$ is a homogeneous matrix with respect to this degree pair.

In order to deform a homogeneous ideal having an $\mathcal{O}$-border basis to its border form ideal, we may try to construct a suitable rational curve inside the homogeneous border basis scheme. If $\mathcal{O}$ has a maxdeg_W border (see Definition 2.7), this plan can be carried out as follows.

Theorem 5.3. (Homogeneous Maxdeg Border Bases)
Suppose that the order ideal $\mathcal{O}$ has a maxdeg_W border.

a) The generic homogeneous multiplication matrices commute.

b) Let $d=\max\{\deg_W(t_1),\dots,\deg_W(t_\mu)\}$, let $r=\#\{t\in\mathcal{O}\mid\deg_W(t)=d\}$, and let $s=\#\{t\in\partial\mathcal{O}\mid\deg_W(t)=d\}$. Then the homogeneous border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{hom}}$ is an affine space of dimension $r\,s$.

c) If $I\subset P$ is a homogeneous ideal which has an $\mathcal{O}$-border basis $G=\{g_1,\dots,g_\nu\}$, then there exists a flat family $K[z]\longrightarrow K[z][x_1,\dots,x_n]/J$ such that $\mathcal{O}$ is a $K[z]$-basis of the right-hand side, such that $J|_{z\mapsto 1}\cong I$, and such that $J|_{z\mapsto 0}\cong(b_1,\dots,b_\nu)$. In fact, the ideal J may be defined by writing $g_j=b_j-\sum_{i=1}^{\mu}c_{ij}t_i$ and replacing $c_{ij}\in K$ by $c_{ij}\,z\in K[z]$ for all i,j.

Proof. To prove claim a), we examine the entry at position (α,β) of a product $\mathcal{A}_k\mathcal{A}_\ell$. Let $\mathcal{A}_k=(a_{ij})$ and $\mathcal{A}_\ell=(a'_{ij})$. We want to examine the element $\sum_{\gamma=1}^{\mu}a_{\alpha\gamma}a'_{\gamma\beta}$. If $a'_{\gamma\beta}\neq 0$, the term t_γ is contained in the support of the representation of $x_\ell\,t_\beta$ in terms of the basis $\mathcal{O}$. Since $(g_1,\dots,g_\nu)$ is a homogeneous ideal with respect to the grading on $K[x_1,\dots,x_n,c_{11},\dots,c_{\mu\nu}]$ defined by the matrix $\overline{W}$ for which $\deg_{\overline{W}}(c_{ij})=0$ and $\deg_{\overline{W}}(x_i)=\deg_W(x_i)=w_i$, we have the relations $\deg_W(t_\gamma)=\deg_W(x_\ell t_\beta)>\deg_W(t_\beta)$. For the same reason, if $a_{\alpha\gamma}\neq 0$ we have the relations $\deg_W(t_\alpha)=\deg_W(x_kt_\gamma)>\deg_W(t_\gamma)$. We deduce the inequality $\deg_W(t_\alpha)>\deg_W(x_\ell t_\beta)$. Hence the assumption that $\mathcal{O}$ has a maxdeg_W border implies $x_\ell t_\beta\notin\partial\mathcal{O}$. We conclude that $x_\ell t_\beta\in\mathcal{O}$, $t_\gamma=x_\ell t_\beta$, and hence $a'_{\gamma\beta}=1$. Therefore, in order to get $a_{\alpha\gamma}a'_{\gamma\beta}\neq 0$ in the sum above, we need to have $a'_{\gamma\beta}=1$ and $t_\gamma=x_\ell t_\beta$. In particular, this condition fixes γ.

If the surviving summand $a_{\alpha\gamma}$ of $\sum_{\gamma=1}^{\mu}a_{\alpha\gamma}a'_{\gamma\beta}$ is not zero, there are two possibilities. Either we have $t_\alpha=x_kt_\gamma$ and thus $a_{\alpha\gamma}=1$, or we have $x_kt_\gamma=b_j$, $t_\alpha\in\mathrm{Supp}(b_j-g_j)$, and hence $a_{\alpha\gamma}=c_{\alpha j}$. In the first case, we have $t_\alpha=x_kx_\ell t_\beta$. In the second case, we have $x_kx_\ell t_\beta=b_j$ and $t_\alpha\in\mathrm{Supp}(b_j-g_j)$. Now it is clear that if we examine the product $\mathcal{A}_\ell\mathcal{A}_k$, we get the same conditions. Therefore we conclude that $\mathcal{A}_k\mathcal{A}_\ell=\mathcal{A}_\ell\mathcal{A}_k$.

Next we show b). The entries of the commutators $\mathcal{A}_k\mathcal{A}_\ell - \mathcal{A}_\ell\mathcal{A}_k$ are the defining equations of the scheme $\mathbb{B}_{\mathcal{O}}^{\rm hom}$ in the affine subspace $\mathcal{Z}(c_{ij} \mid \deg_W(t_i) \neq \deg_W(b_j))$ of $\mathbb{A}^{\mu\nu}$. By a), these commutators are all zero. The number $r\,s$ is precisely the dimension of this affine subspace.

To show c), it now suffices to connect the given point in this affine space by a line to the origin and to apply Corollary 3.5. $\square$

If an ideal I has an $\mathcal{O}$-border basis and $\mathcal{O}$ has a $\max\deg_W$ border for some grading given by a matrix $W \in \mathrm{Mat}_{1,n}(\mathbb{N}_+)$, we can combine the two flat families of Theorem 2.4 and part c) of the theorem above. As an illustration, we continue the discussion of Example 2.9.

Example 5.4. Let I be the ideal $I = (x^2 + xy - \frac{1}{2}y^2 - x - \frac{1}{2}y,\, y^3 - y,\, xy^2 - xy)$ in $K[x,y]$, where $\mathrm{char}(K) \neq 2$, and let $\mathcal{O} = \{1, x, x^2, y, y^2\} \subset \mathbb{T}^2$. Using the fact that $\mathcal{O}$ has a $\max\deg_W$ border with respect to the standard grading, we have already deformed I to $\mathrm{DF}_W(I) = (x^3,\, x^2y,\, xy + x^2 - \frac{1}{2}y^2,\, xy^2,\, y^3)$.

Now we apply the theorem. We equip the summands x^2 and y^2 in the third polynomial with a factor z and get $J = (x^3,\, x^2y,\, xy + zx^2 - \frac{1}{2}zy^2,\, xy^2, y^3)$. As we now let $z \longrightarrow 0$, we get the border form ideal of I. This is a flat deformation by part c) of the theorem. We can also directly check that the multiplication matrices

$$\mathcal{A}_x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -z & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}z & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{A}_y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -z & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}z & 0 & 1 & 0 \end{pmatrix}$$

commute as elements of $\mathrm{Mat}_5(K[z])$.

Notice that, at least following the approach taken here, it is not possible to connect I to $\mathrm{BT}_{\mathcal{O}}$ using just one irreducible rational curve on the border basis scheme. The next example shows that the maxdeg border property is indispensable for the theorem to hold.

Example 5.5. The order ideal $\mathcal{O} = \{1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2\} \subseteq \mathbb{T}^2$. does not have a $\max\deg_W$ border with respect to any grading given by a matrix $W \in \mathrm{Mat}_{1,2}(\mathbb{N}_+)$. The generic homogeneous $\mathcal{O}$-border basis for the standard grading is $G = \{g_1, \ldots, g_6\}$ with $g_1 = y^3 - c_{71}x^2y - c_{81}xy^2$, $g_2 = x^3 - c_{72}x^2y - c_{82}xy^2$, $g_3 = xy^3 - c_{93}x^2y^2$, $g_4 = x^3y - c_{94}x^2y^2$, $g_5 = x^2y^3$, and $g_6 = x^3y^2$.

For the defining ideal of $\mathbb{B}_{\mathcal{O}}^{\rm hom}$, we find $(c_{82}c_{93} + c_{72} - c_{94},\, c_{71}c_{94} + c_{81} - c_{93})$. Hence $\mathbb{B}_{\mathcal{O}}^{\rm hom}$ is not a 2-dimensional affine space (as would be the case if the theorem were applicable), but isomorphic to a 4-dimensional affine space via the projection to $\mathcal{Z}(c_{72}, c_{81})$.

Another consequence of the theorem is that the homogeneous border basis scheme can have a dimension which is higher than $n\mu$, the natural generalization of the dimension of $\mathbb{B}_{\mathcal{O}}$ for $n = 2$ (see Remark 3.2).

Example 5.6. (Iarrobino) In the paper [11] Iarrobino proves that Hilbert schemes need not be irreducible (see also [19], Theorem 18.32). In particular, he produces an example which can easily be explained using homogeneous border basis schemes. Let $\mathcal{O}$ be an order ideal in $\mathbb{T}^3$ consisting of all terms of degree ≤ 6 and 18 terms of degree seven. Then we have $d = 7$ and $r = s = 18$ in part b) of the theorem.

Hence $\mathbb{B}_{\mathcal{O}}^{\rm hom}$ is isomorphic to an affine space of dimension 324. In particular, it follows that $\dim(\mathbb{B}_{\mathcal{O}}) \ge 324$. On the other hand, the irreducible component of $\mathbb{B}_{\mathcal{O}}$ containing the points corresponding to reduced ideals has dimension $3\cdot\mu = 3\cdot 102 = 306$.

In the maxdeg border case, we can also compare the dimension of $\mathbb{B}_{\mathcal{O}}^{\rm hom}$ to the dimension of the *zero fiber* Z, i.e. the dimension of the subscheme of $\mathbb{B}_{\mathcal{O}}$ parametrizing schemes supported at the origin. Since $\mathbb{B}_{\mathcal{O}}^{\rm hom}$ is contained in Z, the preceding example implies that the dimension of Z can be larger than $n\mu$, the dimension of the irreducible component of $\mathbb{B}_{\mathcal{O}}$ containing the points corresponding to reduced ideals. For $n=2$, a more precise estimate is available.

Example 5.7. Let $n=2$. Then the dimension of Z is $\mu-1$ by [2]. If $\mathcal{O}$ has a maxdeg border then the theorem yields $s = d+1-r$ and $\dim(\mathbb{B}_{\mathcal{O}}^{\rm hom}) = r(d+1-r) \le (\frac{d+1}{2})^2$. This agrees with $\mathbb{B}_{\mathcal{O}}^{\rm hom} \subseteq Z$ since $(\frac{d+1}{2})^2 \le \frac{d(d+1)}{2} + r - 1 = \mu - 1$.

Let us end this section with an example application of Theorem 5.3.

Example 5.8. In [19], Example 18.9, the authors consider the ideal $I = (x^2 - xy,\; y^2 - xy,\; x^2y,\; xy^2)$ in the ring $\mathbb{C}[x,y]$. It has a border basis with respect to the order ideal $\mathcal{O} = \{1, x, y, xy\}$, i.e. it corresponds to a point in $\mathbb{B}_{\mathcal{O}}$. It is clear that no matter which term ordering σ one chooses, it is not possible to get $\mathcal{O}_\sigma(I) = \mathcal{O}$, since $x^2 >_\sigma xy$ implies $xy >_\sigma y^2$, and therefore $xy \notin \mathcal{O}_\sigma(I)$. The consequence is that if one wants to connect I to a monomial ideal in the Hilbert scheme, the deformation to $\mathrm{LT}_\sigma(I)$ with respect to any term ordering σ leads to a monomial ideal which is not (x^2, y^2), i.e. not in $\mathbb{B}_{\mathcal{O}}$.

On the other hand, by Example 3.8, we know that it is possible to deform the ideal I to (x^2, y^2). But we can do even better: since the ideal I is homogeneous, it belongs to the family parametrized by the homogeneous border basis scheme $\mathbb{B}_{\mathcal{O}}^{\rm hom}$ which is an affine space of dimension $r\,s = 1\cdot 2 = 2$ by Theorem 5.3. The full family of homogeneous ideals is $(x^2 - zaxy,\; y^2 - zbxy,\; x^2y,\; xy^2)$. Putting $a = b = 1$, we get

$$\Phi : K[z] \longrightarrow \mathbb{C}[x,y,z]/(x^2 - zxy,\, y^2 - zxy,\, x^2y,\, xy^2)$$

the desired flat deformation.

Acknowledgements. Part of this work was conducted during the Special Semester on Gröbner Bases, February 1 to July 31, 2006, organized by the RICAM Institute (Austrian Academy of Sciences) and the RISC Institute (Johannes Kepler University) in Linz, Austria. The authors thank these institutions, and in particular Prof. Bruno Buchberger, for the warm hospitality and the excellent work atmosphere they experienced in Linz.

References

[1] J. Abbott, C. Fassino and M. Torrente, Notes on stable border bases, `arXiv:math\0706.2316v1`.

[2] J. Briançon, Description de $\mathrm{Hilb}^n C\{x,y\}$, Invent. Math. **41** (1977), 45–89.

[3] The CoCoA Team, CoCoA*: a system for doing Computations in Commutative Algebra*, available at `http://cocoa.dima.unige.it`.

[4] A. Conca and G. Valla, Canonical Hilbert-Burch matrices for ideals of $k[x,y]$, `arXiv:math\0708.3576`.

[5] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer, Heidelberg 1995.

[6] T.S. Gustavsen, D. Laksov and R.M. Skjelnes, An elementary, explicit proof of the existence of Hilbert schemes of points, `arXiv:math\0506.161v1`.

[7] M. Haiman, q,t-Catalan numbers and the Hilbert scheme, Discr. Math. **193** (1998), 201–224.

[8] D. Heldt, M. Kreuzer, S. Pokutta, and H. Poulisse, Approximate computation of zero-dimensional ideals, preprint 2006.

[9] M. Huibregtse, A description of certain affine open schemes that form an open covering of $\mathrm{Hilb}^n_{\mathbb{A}^2_k}$, Pacific J. Math. **204** (2002), 97–143.

[10] M. Huibregtse, An elementary construction of the multigraded Hilbert scheme of points, Pacific J. Math. **223** (2006), 269–315.

[11] A. Iarrobino, Reducibility of the families of 0-dimensional schemes on a variety, Invent. Math. **15** (1972), 72–77.

[12] S. Katz, The desingularization of $\mathrm{Hilb}^4\mathbb{P}^3$ and its Betti numbers, in: F. Orecchia and L. Chiantini (eds.), *Zero-dimensional schemes (Ravello, 1992)*, de Gruyter, Berlin 1994, 23–242.

[13] A. Kehrein and M. Kreuzer, Characterizations of border bases, J. Pure Appl. Alg. **196** (2005), 251–270.

[14] A. Kehrein and M. Kreuzer, Computing border bases, J. Pure Appl. Alg. **205** (2006), 279–295.

[15] A. Kehrein, M. Kreuzer and L. Robbiano, An algebraist's view on border bases, in: A. Dickenstein and I. Emiris (eds.), *Solving Polynomial Equations: Foundations, Algorithms, and Applications*, Springer, Heidelberg 2005, 169–202.

[16] M. Kreuzer and L. Robbiano, *Computational Commutative Algebra 1*, Springer, Heidelberg 2000.

[17] M. Kreuzer and L. Robbiano, *Computational Commutative Algebra 2*, Springer, Heidelberg 2005.

[18] K. Lee, On the symmetric subscheme of Hilbert scheme of points, `arXiv:math\0708.3390`.

[19] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, Springer, New York 2005.

[20] B. Mourrain, A new criterion for normal form algorithms, AAECC Lecture Notes in Computer Science **1719** (1999), 430–443.

[21] R. Notari and M. L. Spreafico, A stratification of Hilbert schemes by initial ideals and applications, Manuscripta Math. **101** (2000), 429–448.

[22] H.J. Stetter, *Numerical Polynomial Algebra*, SIAM, Philadephia 2004.

Fakultät für Informatik und Mathematik, Universität Passau, D-94030 Passau, Germany

E-mail address: `kreuzer@uni-passau.de`

Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, I-16146 Genova, Italy

E-mail address: `robbiano@dima.unige.it`