Computing the best approximation from a set of scaled affine combinations

Hennie Poulisse

Abstract

Explicit computations are presented to calculate in a real inner product space the metric projection on - a translate of - a finitely generated convex cone. This result is then used in the Boyle-Dijkstra Theorem to compute in this setting the best approximation from a set of scaled affine combinations.

1 Introduction

We start by formulating the problem addressed in this paper, followed by the motivation to investigate it.
A computable solution is presented for the following problem:

Given a real inner product space $\mathbb{X}$ *with inner product* $\langle \cdot\,, \cdot \rangle : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$*, given a finite set of linearly independent elements of* $\mathbb{X}$

$$\mathbb{Y} = \{ \, y_1, \ldots, y_n \, \} \tag{1}$$

and given the collections of real numbers $\alpha_1, \ldots, \alpha_n$ *and* $\beta_1, \ldots, \beta_n$ *with* $\alpha_i < \beta_i$ *and given the set*

$$\Gamma = \{ \, \chi \in \mathbb{R}^n \mid (\chi)_i \in [\alpha_i, \beta_i] \; , \; i = 1, \ldots, n \, \} \tag{2}$$

with $[\alpha_i, \beta_i]$ *a closed interval in* $\mathbb{R}$*, find the best approximation* - see definition 2 below - *to an element* $x \in \mathbb{X}$ *from the set*

$$\mathbb{H} = \{ \, y \in \mathbb{X} \mid y = \sum_{i=1}^{n} \gamma_i y_i \; , \; y_i \in \mathbb{Y} \; , \; \gamma \in \Gamma \; , \; \gamma_i = (\gamma)_i \, \} \tag{3}$$

The element $x \in \mathbb{X}$*, the space* $\mathbb{X}$*, and the sets* $\mathbb{Y}\,, \mathbb{H}\,, \Gamma$ *and the real numbers* $\{\alpha_i\}$ *and* $\{\beta_i\}$ *are fixed throughout this paper, and so we avoid unnecessary repetitions of these objects in the different statements that follow.*

To place the definition of the set $\mathbb{H}$ in perspective, recall the following definition - see [2], [6] or [8]:

Definition 1 *Let $\mathbb{A}$ be a non-empty subset of the inner product space $\mathbb{X}$. The affine hull $aff(\mathbb{A})$ of $\mathbb{A}$ is the set of all affine combinations of $\mathbb{A}$:*

$$aff(\mathbb{A}) = \{ y \in \mathbb{X} \mid y = \sum_{i=1}^{m} \tau_i a_i \, , \, a_i \in \mathbb{A} \, , \, \tau_i \in \mathbb{R} \, , \, \sum_{i=1}^{m} \tau_i = 1 \, , \, m \in \mathbb{N} \}$$

$y \in \mathbb{H}$ can be written in the following way: $y = \gamma_\sigma \sum_{i=1}^{n} \bar{\gamma}_i y_i$ ($y_i \in \mathbb{Y} \, , \gamma = (\gamma_i) \in \Gamma \, , \, \gamma_\sigma = \sum_i^n \gamma_i \, , \, (\bar{\gamma}_i) = (\gamma_i / \gamma_\sigma)$). Hence

$$\mathbb{H} \subset \Lambda aff(\mathbb{Y})$$

where $\Lambda = [\sum_{i=1}^{n} \alpha_i, \sum_{i=1}^{n} \beta_i] \subset \mathbb{R}$, i.e. $\mathbb{H}$ is a set of scaled, affine combinations. In particular $\mathbb{H}$ is a *closed, convex* subset of $\mathbb{X}$.

Following F. Deutsch [6], the best approximation is defined in the following way:

Definition 2 *Let $\mathbb{A}$ be a nonempty subset of the inner product space $\mathbb{X}$, and let $x \in \mathbb{X}$. An element $a_0 \in \mathbb{A}$ is called a best approximation to x from $\mathbb{A}$ if $\|x - a_0\| = \inf_{a \in \mathbb{A}} \|x - a\|$, where $\| \cdot \|$ is the norm induced by the inner product on $\mathbb{X}$.*
The set of all best approximations to $x \in \mathbb{X}$ from $\mathbb{A}$ is denoted by $\mathbf{P}_{\mathbb{A}}(x)$. The mapping $\mathbf{P}_{\mathbb{A}}$ from $\mathbb{X}$ into the subsets of $\mathbb{A}$ is called the metric projection onto $\mathbb{A}$. If each $x \in \mathbb{X}$ has exactly one best approximation in $\mathbb{A}$, i.e. $\mathbf{P}_{\mathbb{A}}(x)$ is a singleton, then $\mathbb{A}$ is called a Chebyshev set.

The problem stated above is important in many technical applications. In [10] an application is given in oil - and gas production operations.
Explicit computations of approximations from *closed convex sets*, including *closed convex cones* are sparse in the literature, as opposed to those from - translates of - subspaces - see F. Deutsch [6] and J-B. Hiriart-Urruty and C. Lemaréchal [8]. Our contribution is that we give an explicit computation of the metric projection on a *finitely generated convex cone*, and this result is used subsequently to compute the best approximation from the set $\mathbb{H}$.
We refer to F. Deutsch [6] and also D. Amir [1] for the importance of choosing the inner product space setting.
The rest of this paper is organized in the following way: in the next section we explore the mathematical setting of our problem with special attention for the linear independence assumption of $\mathbb{Y}$ in (1), this is then followed by an explicit computation of the metric projection on - a translate of - a finitely generated convex cone, and finally in the last section we compute the best approximation from the convex set $\mathbb{H}$.
Continuing a line of argument until a computable solution is clearly 'visible' is pursued throughout this paper.

2 A Mathematical Inventory

To find the best approximation from $\mathbb{H}$ we have to reveal the geometric structure of our problem description. A good start in this respect is the observation that the convexity of $\mathbb{H}$ means that it is equivalent to its *convex hull* $co(\mathbb{H})$ - see [6]:

$$\begin{aligned} \mathbb{H} &= co(\mathbb{H}) \\ &:= \{ y \in \mathbb{X} \mid y = \sum_{i=1}^{m} \tau_i h_i \, , \, h_i \in \mathbb{H} \, , \, \tau_i \in \mathbb{R}^+ \, , \, \sum_{i=1}^{m} \tau_i = 1 \, , \, m \in \mathbb{N} \} \end{aligned} \tag{4}$$

where $\mathbb{R}^+$ denotes the set of non-negative real numbers. The crucial role played by the set $\mathbb{Y}$ from equation (1) is substantiated in the following result, which is the famous Carathéodory theorem from $\mathbb{R}^n$ - see [2] or [8] - in our setting.

Theorem 1 *Every element $y \in \mathbb{H}$ can be written as a convex combination of $n+1$ elements $h_1, \ldots, h_{n+1}$ from $\mathbb{H}$:*

$$y = \sum_{i=1}^{n+1} \nu_i h_i \, , \quad \sum_{i=1}^{n+1} \nu_i = 1, \quad \nu_i \geq 0 \tag{5}$$

Proof: Every $y \in \mathbb{H}$ can be written as a convex combination $y = \sum_{i=1}^{m} \nu_i h_i$ of some elements $h_1, \ldots, h_m$ of $\mathbb{H}$. We can assume that $\nu_i > 0$ for all $i = 1, \ldots, m$. If $m < n+1$, we can always add terms $0 h_i$ to get a convex combination with $n+1$ terms. Now suppose that $m > n+1$. Consider the following equations in the real variables $\kappa_1, \ldots, \kappa_m$:

$$\sum_{i=1}^{m} \kappa_i h_i = 0 \quad \& \quad \sum_{i=1}^{m} \kappa_i = 0 \tag{6}$$

Using equation (3) each h_i has the following representation in terms of the $y_i \in \mathbb{Y}$:

$$h_i = \sum_{j=1}^{n} \gamma_j^i y_j \quad (\gamma^i \in \Gamma \, , \, (\gamma^i)_j = \gamma_j^i \, , \, i \in \{1, \ldots, m\} \,) \tag{7}$$

Because the set $\mathbb{Y}$ is linearly independent, equations (6) are equivalent to the following system of linear equations in $\kappa_1, \ldots, \kappa_m$:

$$\begin{pmatrix} \gamma_1^1 & \cdots & \gamma_1^m \\ \cdots & \cdots & \cdots \\ \gamma_n^1 & \cdots & \gamma_n^m \\ 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \cdots \\ \kappa_{m-1} \\ \kappa_m \end{pmatrix} = 0 \tag{8}$$

From this moment on our proof follows the one given by A. Barvinok [2] in a $\mathbb{R}^n$ setting; we present the details for easy reference. Because $m > n+1$ there must be a non-trivial solution $\kappa_1, \ldots, \kappa_m$, and because $\sum_{i=1}^{m} \kappa_i = 0$, some κ_i are strictly positive, and some are strictly negative. Let

$$\mu = \min\{\nu_i/\kappa_i \mid \kappa_i > 0\} = \nu_{i_0}/\kappa_{i_0}$$

Define $\tilde{\nu}_i = \nu_i - \mu\kappa_i \, (i = 1, \ldots, m) \Rightarrow \tilde{\nu}_i \geq 0 \, \forall \, i$ and $\tilde{\nu}_{i_0} = 0$, and we compute $\sum_{i=1}^{m} \tilde{\nu}_i = \sum_{i=1}^{m} \nu_i - \mu \sum_{i=1}^{m} \kappa_i = 1$, and $\sum_{i=1}^{m} \tilde{\nu}_i h_i = \sum_{i=1}^{m} \nu_i h_i - \mu \sum_{i=1}^{m} \kappa_i h_i = y$. So apparently we can write y as a convex combination of $m-1$ rather than m points - h_{i_0} is omitted. Iterating this procedure, we obtain y as a convex combination of $n+1$ or fewer points. $\square$

Now our goal is to come to *explicit* calculations of the metric projection onto $\mathbb{H}$, and this means that sooner or later we have to resort to a basis for $\mathbb{H}$. The problem with the previous result is that it does not provide a basis for $\mathbb{H}$. So maybe we can find an environment of $\mathbb{H}$ that offers a sufficiently rich relative topology for $\mathbb{H}$. The most obvious choice in this respect would seem the ambient space $\mathbb{X}$ of $\mathbb{H}$. But although $\mathbb{X}$ provides its extremely powerful inner product plus all geometric ramifications associated with it, it does not qualify as a convenient 'working' environment for $\mathbb{H}$ because it is not specific enough. Therefore it may be a better idea to start at the other end, i.e. starting from $\mathbb{H}$. And then a natural way to proceed would seem lifting the restrictions on the coefficients in the convex hull representation of $\mathbb{H}$ in equation (4). If we leave out the restriction that the coefficients must sum up to 1, we arrive at the *conical hull* $con(\mathbb{H})$ of $\mathbb{H}$:

$$con(\mathbb{H}) = \{y \in \mathbb{X} \mid y = \sum_{i=1}^{m} \tau_i h_i \, , \, h_i \in \mathbb{H} \, , \, \tau_i \in \mathbb{R}^+ \, , \, m \in \mathbb{N}\} \tag{9}$$

To give an interpretation of $con(\mathbb{H})$, we need the following definition:

Definition 3 *A subset $\mathbb{C}$ of $\mathbb{X}$ is called a convex cone if $\rho y + \tau z \in \mathbb{C}$ whenever $y, z \in \mathbb{C}$ and $\rho, \tau \in \mathbb{R}^+$*

$con(\mathbb{H})$ is the intersection of all convex cones that contain $\mathbb{H}$ - see [6] or [8]. In convex analysis convex cones are intermediate between subspaces and general convex sets, and that means that they tend to have properties that are somewhat closer to the more favorable properties of subspaces than those of general convex sets. Our first observation seems promising in this respect: from (9) it follows that $0 \in con(\mathbb{H})$. Because we did not assume that $0 \in \Gamma$, it is unknown whether $\mathbb{H}$ contains 0. This is immediately an improvement of $con(\mathbb{H})$ over $\mathbb{H}$, for in view of our ambition to come up with an explicit calculation of the metric projection onto $\mathbb{H}$, we will doubtlessly need the 'reference point' 0. The improvement is however not sufficient for $con(\mathbb{H})$ to qualify as working environment for $\mathbb{H}$, since on comparing equations (4) and (9) it follows that

$$con(\mathbb{H}) = \mathbb{R}^+ co(\mathbb{H}) = \mathbb{R}^+ \mathbb{H} \tag{10}$$

So basically $con(\mathbb{H})$ is a scaled version of $\mathbb{H}$, and hence from our perspective this does not give us what we are looking for. Lifting instead the other restriction on the coefficients in (4) gives yet another hull of $\mathbb{H}$, namely the *affine hull* $aff(\mathbb{H})$ of $\mathbb{H}$, the set of all affine combinations of elements of $\mathbb{H}$ - see definition 1. If we compare $aff(\mathbb{H})$ and $con(\mathbb{H})$ it follows that we may have lost our reference point 0 again. To assess the importance of this observation, we need the following definition:

Definition 4 *Let $\mathbb{A}$ be a non-empty subset of $\mathbb{X}$. The subspace spanned by $\mathbb{A}$ denoted by $span(\mathbb{A})$ is the set of all finite linear combinations of elements of $\mathbb{A}$, i.e.*

$$span(\mathbb{A}) = \{y \in \mathbb{X} \mid y = \sum_{i=1}^{m} \tau_i a_i \, , \, a_i \in \mathbb{A} \, , \, \tau_i \in \mathbb{R} \, , \, m \in \mathbb{N} \}$$

If $\mathbb{A}$ consists of a finite number of elements, i.e. $\mathbb{A} = \{a_1, \ldots, a_k\}$, $span(\mathbb{A})$ is finite dimensional, and a maximal, linearly independent subset of $\{a_1, \ldots, a_k\}$ is a basis of $span(\mathbb{A})$.

$span(\mathbb{H})$ is the smallest subspace of $\mathbb{X}$ that contains $\mathbb{H}$. Note by the way that $span(\mathbb{H})$ evolves from $co(\mathbb{H})$ in equation (4) by lifting *all* restrictions on the real coefficients. With respect to ordering by set inclusion $aff(\mathbb{H})$ is closer to $\mathbb{H}$ than $span(\mathbb{H})$ because of the additional requirement on the coefficients in $aff(\mathbb{H})$. On the other hand if $0 \in \mathbb{H}$, then $aff(\mathbb{H}) = span(\mathbb{H})$ - see R. Webster [11] or F. Deutsch [6]. Hence

$$span(\mathbb{H}) = aff(\mathbb{H} \cup \{0\}) \tag{11}$$

The following proposition relates this $\mathbb{H}$ environment to data from our problem description:

Proposition 1

$$aff(\mathbb{H} \cup \{0\}) = span(\mathbb{H}) = span(\mathbb{Y})$$

Proof: $\mathbb{H} \subset \Lambda aff(\mathbb{Y}) \Rightarrow span(\mathbb{H}) \subset span(\mathbb{Y})$. Conversely let $y = \sum_{i=1}^{n} \tau_i y_i \in span(\mathbb{Y})$. Choose n linearly independent $\mathbb{R}^n$ vectors $\gamma^j \in \Gamma$ $(j = 1, \ldots, n)$. Clearly this is always possible since $\alpha_i < \beta_i$ (cf. equation (2)). Then $y = \sum_{i=1}^{n} \theta_i h_i \in span(\mathbb{H})$ where $\theta = (\theta_i) \in \mathbb{R}^n$ is the unique solution of the following system of linear equations:

$$\begin{pmatrix} \gamma_1^1 & \cdots & \gamma_1^n \\ \ldots & \ldots & \ldots \\ \gamma_n^1 & \cdots & \gamma_n^n \end{pmatrix} \begin{pmatrix} \theta_1 \\ \ldots \\ \theta_n \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \ldots \\ \tau_n \end{pmatrix}$$

Hence $span(\mathbb{Y}) \subset span(\mathbb{H})$ $\square$

So we finally settle for the finite dimensional subspace $\mathbb{S} = span(\mathbb{Y})$ as our convenient working environment for $\mathbb{H}$, because it is the object in $\mathbb{X}$ closest to

$\mathbb{H}$ that provides a known basis.

We now digress a little to discuss a topic that is in particular of importance in applications. After all, claiming a 'computable' solution for the metric projection onto $\mathbb{H}$ for 'clean' situations only, may not be that convincing. The point here is that dropping the linear independence assumption on $\mathbb{Y}$ in (1) may lead to inconsistencies in our problem setting - see subsection 2.1. Specifically in applications where the y_i may result from some estimation procedure, linear dependence of the y_i is likely to occur. There may be practical reasons in trying to express the to be approximated element $x \in \mathbb{X}$ in principle in *all* y_i despite linear dependence among them. Therefore we discuss how one could deal with this situation in our current setting. We would like to add that a more elegant way than the one presented here to deal with a possible linear independence in the family $\{y_i\}$ is to note that they are *always* a *frame* for $\mathbb{S}$ - see O. Christensen [5]. In particular the inconsistencies referred to above would then be avoided; the results following this route will be published separately. We note that this type of linear dependence problems is treated in the literature on convex analysis along our current lines, but mostly in a $\mathbb{R}^n$ setting where it has less consequences.

2.1 Linear Dependence

The results in this subsection depend on the dimension of $\mathbb{S} = span(\mathbb{Y})$*; this is emphasized through the notation used in this subsection, in that all relevant objects have appropriate super - and/or sub indices related to* $dim(\mathbb{S})$*. Specifically* $\mathbb{Y}, \Gamma,$ *and* $\mathbb{H}$ *from equations* (1) - (3) *are in this subsection denoted by, respectively* $\mathbb{Y}_n, \Gamma_n,$ *and* $\mathbb{H}_n$*.*
Suppose

$$dim(\mathbb{S}) = m < n$$

- A good way to check the dimension of $\mathbb{S}$ is to construct through the *Gram-Schmidt orthogonalization process* - see e.g. [6] - the maximal orthonormal set from $\mathbb{Y}_n$. - Then only m out of the n elements of $\mathbb{Y}_n$ constitute a basis for $\mathbb{S}$. Without loss of generality we may assume the first m elements to be a basis :

$$\mathbb{Y}_m = \{\, y_i \in \mathbb{Y}_n \mid i \in \{1, \dots, m\}\,\} \tag{12}$$

We can express the elements of $\mathbb{Y}_n \setminus \mathbb{Y}_m$ in those of $\mathbb{Y}_m$:

$$y_j = \zeta_1^j y_1 + \cdots + \zeta_m^j y_m \quad (\, j = m+1, \dots, n \;,\; \zeta_i^j \in \mathbb{R}\,) \tag{13}$$

The ζ_i^j in equation (13) depend on $\mathbb{Y}_m \cup \{y_j\}$ via the *normal equations* - see F. Deutsch [6]:

$$\mathbf{G}(y_1, \dots, y_m)\zeta^j = \theta^j \quad (\, j = m+1, \dots, n\,) \tag{14}$$

where the elements of $m \times m$ *Gram matrix* $\mathbf{G}(y_1, \ldots, y_m)$ and the elements of the $\mathbb{R}^m$ vectors ζ^j and θ^j are given by respectively

$$\begin{aligned} (\mathbf{G}(y_1, \ldots, y_m))_{i,k} &= \langle\, y_k, y_i \,\rangle \quad (\, i, k = 1, \ldots, m\,) \\ (\zeta^j)_i &= \zeta_i^j \qquad\quad\;\; (\, i = 1, \ldots, m\,) \\ (\theta^j)_i &= \langle\, y_j, y_i \,\rangle \quad (\, i = 1, \ldots, m\,) \end{aligned} \tag{15}$$

Because $\mathbb{Y}_m$ is linearly independent $\det(\mathbf{G}(y_1, \ldots, y_m)) > 0$ - see e.g. [6] - and so equation (14) gives a unique solution $\zeta^j \; \forall j$. Furthermore

$$y_j \neq 0 \;\Rightarrow\; \theta^j \neq 0 \;\Rightarrow\; \textit{at least one of the } \zeta_i^j \neq 0 \;\forall\, j \tag{16}$$

If we would not have made the linear independence assumption in (1), then from equations (3) and (13) it follows that for some $\gamma \in \Gamma_n$ we have $\sum_{i=1}^n \gamma_i y_i = \sum_{i=1}^m (\gamma_i + \sum_{j=m+1}^n \gamma_j \zeta_i^j) y_i$ and so $\sum_{i=1}^n \gamma_i y_i \in \mathbb{H}_n$, whereas we may very well have that $\sum_{i=1}^m (\gamma_i + \sum_{j=m+1}^n \gamma_j \zeta_i^j) y_i \notin \mathbb{H}_n$, first of all because we did *not* assume that $0 \in \Gamma_n$, and secondly, even if this would be the case, there is no guarantee that $(\gamma_i + \sum_{j=m+1}^n \gamma_j \zeta_i^j) \in [\alpha_i, \beta_i]$ since $\mathbb{X}$ and hence also $\mathbb{Y}_n$ are completely arbitrary. Such a situation, resulting from the fact that $\mathbb{H}_n$ is a set but not a subspace, is of course absurd.
We now describe a proper way to handle this case in our setting. For $i = 1, \ldots, m$ consider the functionals $f_i : \mathbb{R}^{n-m} \to \mathbb{R}$ and moreover the sets Γ_n^{n-m} and Γ_n^m defined by, respectively

$$\begin{aligned} f_i(\chi) &= \sum_{j=1}^{n-m} \zeta_i^{j+m} \chi_j \\ \Gamma_n^{n-m} &= \{\chi \in \mathbb{R}^{n-m} \mid (\chi)_i \in [\alpha_{i+m}, \beta_{i+m}]\,,\; i = 1, \ldots, n-m\,\} \\ \Gamma_n^m &= \{\chi \in \mathbb{R}^m \mid (\chi)_i \in [\alpha_i, \beta_i]\,,\; i = 1, \ldots, m\,\} \end{aligned} \tag{17}$$

Define

$$\begin{aligned} \{1, \ldots, m\} &= I_{m_1} \cup I_{m_2} \\ |I_{m_1}| + |I_{m_2}| &= m_1 + m_2 = m \quad (\, 0 < m_1 \leq m\,) \\ i &\in \begin{cases} I_{m_1} & \textit{if } \zeta_i^{j+m} \neq 0 \textit{ for some } j \in \{1, \ldots, n-m\} \\ I_{m_2} & \textit{otherwise} \end{cases} \end{aligned} \tag{18}$$

with $|\cdot|$ referring to the number of elements. Recall the following definition:

Definition 5 *Let* $(\cdot\,, \cdot) : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ *be the inner product in* $\mathbb{R}^k$ *for some* $k \geq 1$. *Let* $\eta_1, \ldots, \eta_l$ *be vectors from* $\mathbb{R}^k$ *and let* $\theta_1, \ldots, \theta_l$ *be numbers. The set* $\Pi = \{\, \chi \in \mathbb{R}^k \mid (\eta_i, \chi) \leq \theta_i \;(\, i = 1, \ldots, l\,)\,\}$ *is called a polyhedron. The i-th inequality from the previous equation is called active for* $\chi \in \Pi$ *if* $(\eta_i, \chi) = \theta_i$.
$\chi^{\mathrm{v}} \in \Pi$ *is a* vertex *of* Π *if for any two points* $\chi_1, \chi_2 \in \Pi$ *such that* $\chi^{\mathrm{v}} = (\chi_1 + \chi_2)/2$ *we must have* $\chi^{\mathrm{v}} = \chi_1 = \chi_2$.

The sets Γ_n^{n-m} and Γ_n^m are polyhedra, for, denoting by $\{\xi_1, \ldots, \xi_{n-m}\}$ the 'canonical' basis in $\mathbb{R}^{n-m}$, where each ξ_i has coordinates $(0, \ldots, 0, 1, 0, \ldots, 0)$ (the "1" in the i^{th} position), the defining polyhedral inequalities for Γ_n^{n-m} are $(\chi, -\xi_i) \leq -\alpha_{i+m}$, $(\chi, \xi_i) \leq \beta_{i+m}$ $(i = 1, \ldots, n-m)$, and likewise for Γ_n^m. The parameter l in definition 5 is $2(n-m)$ for Γ_n^{n-m} and $2m$ for Γ_n^m. For $i \in I_{m_1}$ the f_i are *linear* functionals. They can be interpreted as describing the deformation in the *'scaled barycentric'* - adapted terminology from [11] and [8] - coefficients γ_i of the elements $y_i \in \mathbb{Y}_m$ in $\mathbb{H}_n$ in equation (3) when they must also cater for the contribution of the elements from $(\mathbb{Y}_n \setminus \mathbb{Y}_m)$.
Of course we would like to know what the maximal 'deformations' are. Clearly the f_i ($i \in I_{m_1}$) assume their unique minimum - and maximum value on the compact, convex set Γ_n^{n-m}:

$$\alpha_i^{min} = \begin{cases} \min\{f_i(\chi) \mid \chi \in \Gamma_n^{n-m}\} & \textit{if} \quad i \in I_{m_1} \\ 0 & \textit{if} \quad i \in I_{m_2} \end{cases} \tag{19}$$

$$\beta_i^{max} = \begin{cases} \max\{f_i(\chi) \mid \chi \in \Gamma_n^{n-m}\} & \textit{if} \quad i \in I_{m_1} \\ 0 & \textit{if} \quad i \in I_{m_2} \end{cases}$$

Intuitively one expects that, under the given conditions, it must be possible to be more specific about these extremes. In this respect we have the following Theorem, the first part of which is stated in A. Barvinok [2] as a corollary to the finite dimensional version of the Krein-Milman Theorem - which is according to J-B. Hiriart-Urruty and C. Lemaréchal [8] due to Hermann Minkowski - whereas the second part follows directly from the fact that for a vertex in a polyhedron at least $n-m$ of the $2(n-m)$ inequalities describing the polyhedron Γ_n^{n-m} are active - see [2] or [8]:

Theorem 2

(1) Let $i \in I_{m_1}$. There exist vertices $\chi_1^{\mathrm{v}}, \chi_2^{\mathrm{v}} \in \Gamma_n^{n-m}$ such that

$$\begin{aligned} \alpha_i^{min} &= f_i(\chi_1^{\mathrm{v}}) \\ \beta_i^{max} &= f_i(\chi_2^{\mathrm{v}}) \end{aligned}$$

(2) For $j = 1, \ldots, n-m$

$$\chi_{1_j}^{\mathrm{v}} = \begin{cases} \alpha_{j+m} & \text{if} \quad \zeta_i^{j+m} \geq 0 \\ \beta_{j+m} & \text{otherwise} \end{cases} \quad , \quad \chi_{2_j}^{\mathrm{v}} = \begin{cases} \beta_{j+m} & \text{if} \quad \zeta_i^{j+m} \geq 0 \\ \alpha_{j+m} & \text{otherwise} \end{cases} \qquad \square$$

Note that we are not interested in describing the solution sets for α_i^{min} and β_i^{max}; the vertices are points of these solution sets. Theorem 2 shows how to calculate the values in (19). Alternatively one could mechanize this computation by casting (19) as *linear programming* problems - see R. Webster [11] for details. We refer to [10] for an example where these calculations are performed in a specific application.

The decomposition of $\mathbb{Y}_n$ in $\mathbb{Y}_m \cup (\mathbb{Y}_n \setminus \mathbb{Y}_m)$ is reflected in the scaled affine combinations of the $\{y_i\}$ in the following way:

$$\begin{aligned} \mathbb{H}_n^m &= \{\, y = \sum_{i=1}^{m} \gamma_i y_i \mid y_i \in \mathbb{Y}_m \subset \mathbb{Y}_n \,,\, \gamma \in \Gamma_n^m \,,\,,\, \gamma_i = (\gamma)_i \,\} \\ \mathbb{H}_n^{n-m} &= \{\, y = \sum_{i=m+1}^{n} \gamma_i y_i \mid y_i \in \mathbb{Y}_n \setminus \mathbb{Y}_m \,,\, \gamma \in \Gamma_n^{n-m} \,,\, \gamma_i = (\gamma)_i \,\} \end{aligned}$$

The set of scaled affine combinations for the current situation

$$\mathbb{H}_m = \{\, y = \sum_{i=1}^{n} \gamma_i y_i \mid y_i \in \mathbb{Y}_n \,,\, \gamma \in \Gamma_n \,,\, \gamma_i = (\gamma)_i \,,\, dim(span(\mathbb{Y}_n)) = m < n \,\}$$

but represented in terms of the linear independent set $\mathbb{Y}_m$ now follows directly:

Proposition 2

$$\begin{aligned} \mathbb{H}_m &= \{\, y = \sum_{i=1}^{m} \gamma_i y_i \mid y_i \in \mathbb{Y}_m \,,\, \gamma \in \Gamma_m \,,\, \gamma_i = (\gamma)_i \,\} \qquad (20) \\ \Gamma_m &= \{ \chi \in \mathbb{R}^m \mid (\chi)_i \in [\alpha_i^{n-m}, \beta_i^{n-m}] \,,\, i = 1, \ldots, m \,\} \end{aligned}$$

$$\alpha_i^{n-m} = \begin{cases} \alpha_i & \text{if } \alpha_i^{min} \geq 0 \\ \alpha_i + \alpha_i^{min} & \text{otherwise} \end{cases} \,,\; \beta_i^{n-m} = \begin{cases} \beta_i & \text{if } \beta_i^{max} \leq 0 \\ \beta_i + \beta_i^{max} & \text{otherwise} \end{cases}$$

Proof: Compute

$$\mathbb{H}_n^m + \mathbb{H}_n^{n-m} = \bigcup \{\, \mathbb{H}_n^m + z \mid z \in \mathbb{H}_n^{n-m} \,\} = \mathbb{H}_m \quad \square$$

Denoting a best approximation to $x \in \mathbb{X}$ from $\mathbb{H}_m$ in equation (20), if such an approximation exists, by $\hat{x}$, it can be represented in the following way:

$$\hat{x} = \sum_{i=1}^{m} \hat{\gamma}_i y_i \quad (\, \hat{\gamma} \in \Gamma_m \,,\, \hat{\gamma}_i = (\hat{\gamma})_i \,,\, y_i \in \mathbb{Y}_m \,) \qquad (21)$$

Casting this result in terms of the 'original' linearly dependent set $\mathbb{Y}_n$ leads to the following under-determined system of m equations in the n unknowns γ_i:

$$\mathbf{A}\gamma = \hat{\gamma} \quad (\gamma \in \Gamma_n) \qquad (22)$$

where the $m \times n$ matrix $\mathbf{A}$ is, as a result of our construction given in equation (13), directly in reduced row-echelon form:

$$\begin{aligned} \mathbf{A} &= (\mathbf{I}_m \, \mathbf{F}) \qquad (23) \\ (\mathbf{F})_{i,j} &= \zeta_i^{j+m} \quad (\, i = 1, \ldots, m \,,\, j = 1, \ldots, n-m \,) \end{aligned}$$

with $\mathbf{I}_m$ the $m \times m$ identity matrix of $\mathbb{R}^m$. In view of equation (18) $\mathbf{F}$ may have m_2 rows with zeros. To focus first on objects having an index from I_{m_1} we introduce the following notation:

$$\begin{aligned}
\tilde{\gamma} &= ((\hat{\gamma})_i \mid i \in I_{m_1}) \quad (\tilde{\gamma} \textit{ an } m_1 \times 1 \textit{ vector}) \\
\tilde{\mathbf{A}} &= (\mathbf{I}_{m_1}\ \tilde{\mathbf{F}}) \\
\tilde{\mathbf{F}} &= ((\mathbf{F})_{i,j} \mid i \in I_{m_1}\,,\ j = 1, \ldots, n-m) \quad (\tilde{\mathbf{F}} \textit{ an } m_1 \times (n-m) \textit{ matrix}) \\
\tilde{\Gamma}_m &= \{\, \chi \in \mathbb{R}^{m_1} \mid (\chi)_i \in [\alpha_i^{n-m}, \beta_i^{n-m}]\,,\ i \in I_{m_1} \,\} \\
\tilde{\Gamma}_n &= \{\, \chi \in \mathbb{R}^{n-m_2} \mid (\chi)_i \in [\alpha_i, \beta_i]\,,\ i \in \{1, \ldots, n\} \setminus I_{m_2} \,\}
\end{aligned} \tag{24}$$

Note that in view of equation (16) all columns of $\tilde{\mathbf{F}}$ contain at least one element different from zero. Instead of equation (22) we now have

$$\tilde{\mathbf{A}}\gamma = \tilde{\gamma} \quad (\gamma \in \tilde{\Gamma}_n\ ; \sum_{i \in \{1,\ldots,n\} \setminus I_{m_2}} \gamma_i y_i = \sum_{i \in I_{m_1}} \tilde{\gamma}_i y_i\ , \gamma_i = (\gamma)_i\ , \tilde{\gamma}_i = (\tilde{\gamma})_i) \tag{25}$$

The *image* $\mathcal{I}(\tilde{\mathbf{A}})$ of $\tilde{\Gamma}_n$ under $\mathbf{A}$ and the *null space* of $\tilde{\mathbf{A}}$, $\mathcal{N}(\tilde{\mathbf{A}})$, are given by, respectively

$$\begin{aligned}
\mathcal{I}(\tilde{\mathbf{A}}) &= \tilde{\mathbf{A}}\tilde{\Gamma}_n = \tilde{\Gamma}_m \\
\mathcal{N}(\tilde{\mathbf{A}}) &= \textit{span}\{\textit{col}\{\mathbf{N}\}\} = \textit{span}\left\{\textit{col}\left\{\begin{pmatrix} -\tilde{\mathbf{F}} \\ \mathbf{I}_{n-m} \end{pmatrix}\right\}\right\}
\end{aligned} \tag{26}$$

where $\textit{col}\{(\cdot)\}$ refers to the set of column vectors. In using the word image, we have followed John Lee's terminology [9] here, where the word *range* would be reserved for $\tilde{\mathbf{A}}\mathbb{R}^{n-m_2}$.

The first equation in (26) results directly from our construction of Γ_m from Γ_n as given in Proposition 2. Since $\tilde{\gamma} \in \tilde{\Gamma}_m = \mathcal{I}(\tilde{\mathbf{A}})$, the linear system (25) is *consistent*, i.e. has a solution for γ - see [3] or [8].

As for the second equation (26) we note that the column vectors of the matrix $\mathbf{N}$ are linearly independent, and hence form a basis for $\mathcal{N}(\tilde{\mathbf{A}})$. If $\gamma_p \in \tilde{\Gamma}_n$ is any particular solution of (25), then the solution to equation (25) is given by the set $(\gamma_p + \mathcal{N}(\tilde{\mathbf{A}})) \cap \tilde{\Gamma}_n$. From the consistency of the linear system (25) it follows that this set is not empty.

Clearly the structure of our problem is coded in $\mathcal{I}(\tilde{\mathbf{A}})$ and $\mathcal{N}(\tilde{\mathbf{A}})$.

So what we would like to do in order to translate our approximation result $\hat{\gamma}$ from equation (21) from the 'deformed' set $\mathbb{H}_m$ into a result from the 'original' set $\mathbb{H}_n$ is calculating the *generalized inverse* - see [3] - of $\tilde{\mathbf{A}}$ in equation (25) acknowledging its prescribed image $\mathcal{I}(\tilde{\mathbf{A}})$ and null space $\mathcal{N}(\tilde{\mathbf{A}})$. For, a part of the solution, that we denote by $\gamma_0^{m_2}$, can be read off directly from equation (22):

$$\gamma_0^{m_2} = \begin{cases} ((\hat{\gamma})_i \mid i \in I_{m_2}) & \textit{if } m_2 > 0 \\ \emptyset & \textit{if } m_2 = 0 \end{cases} \tag{27}$$

whereas for the 'rest' of the solution we would have γ in equation (25) expressed in terms of $\tilde{\gamma}$, while at the same time this representation honors the structure of our problem. Representations for the generalized inverse of $\tilde{\mathbf{A}}$ with prescribed *image* and null space are *not* available, but they are for prescribed *range* and null space - see A. Ben-Israel and T. Greville [3]. And in that case the generalized inverse is the *Moore-Penrose* - or *pseudo* inverse; the solution associated with this inverse is the *unique minimum norm* solution. The minimum norm solution of equation (25), denoted by $\gamma_0^{n-m_2}$, is characterized in the following way: denote the *set of solutions* to $\min\{\left\|\tilde{\mathbf{A}}\gamma-\tilde{\gamma}\right\|_{2,m_1} \mid \gamma \in \tilde{\Gamma}_n \,; \sum_{i\in\{1,\ldots,n\}\setminus I_{m_2}} \gamma_i y_i = \sum_{i\in I_{m_1}} \tilde{\gamma}_i y_i\}$ by:

$$\Delta = Argmin\{\left\|\tilde{\mathbf{A}}\gamma-\tilde{\gamma}\right\|_{2,m_1} \mid \gamma \in \tilde{\Gamma}_n \,; \sum_{i\in\{1,\ldots,n\}\setminus I_{m_2}} \gamma_i y_i = \sum_{i\in I_{m_1}} \tilde{\gamma}_i y_i\} \tag{28}$$

The set $\Delta \subset \mathbb{R}^{n-m_2}$ is closed and convex, and hence has a unique element of minimal norm:

$$\gamma_0^{n-m_2} \in \Delta \,,\, \left\|\gamma_0^{n-m_2}\right\|_{2,n-m_2} \leq \|\gamma\|_{2,n-m_2} \;\forall\, \gamma \in \Delta \tag{29}$$

The actual calculations to obtain $\gamma_0^{n-m_2}$ can be performed in several ways; we refer to Å. Björck [4] for an overview in the realm of least squares solutions, and to [10] where such calculations are performed in a specific application. True, using equations (28) and (29) we have not obtained a representation for the pseudo inverse of $\tilde{\mathbf{A}}$, but we did obtain the solution associated with it.
Writing

$$\gamma_0 = \begin{pmatrix} \gamma_0^{m_2} \\ \gamma_0^{n-m_2} \end{pmatrix} \tag{30}$$

and denoting its reordering according to $\{1,\ldots,n\}$ also by γ_0, the approximate solution of the best approximation of $x \in \mathbb{X}$ to $\mathbb{H}_m$ - equation (20) - in terms of the elements of the linearly dependent set $\mathbb{Y}_n$ reads as follows:

$$\hat{x} = \sum_{i=1}^{n} \gamma_{0_i} y_i \quad (\, \gamma_{0_i} = (\gamma_0)_i \,) \tag{31}$$

For the sequel of this paper we assume that $\mathbb{Y}$ is linearly independent and hence we suppress again all sub - and super indices related to $dim(\mathbb{S})$.

2.2 Exploring the Setting

That $\mathbb{S} = span(\mathbb{Y})$ is really a convenient environment for $\mathbb{H}$ specifically in relation with finding the metric projection onto it, is substantiated in the following result taken from F. Deutsch [6]:

Theorem 3

(1) Every closed, convex subset of $\mathbb{S}$ *is Chebyshev.*

(2) $\mathbb{S}$ *itself is Chebyshev.* $\square$

So in particular $\mathbb{H}$ is Chebyshev, i.e. our problem stated in section 1 has a unique solution.
Before giving a convenient characterization of metric projections - see also definition 2 - onto a convex set in an inner product space, we need the following definition - [6] or [8]:

Definition 6 *Let* $\mathbb{A}$ *be a nonempty subset of* $\mathbb{X}$*. The dual cone of* $\mathbb{A}$ *is the set*

$$\mathbb{A}^0 = \{\, x \in \mathbb{X} \mid \langle x, a\rangle \leq 0 \quad \forall a \in \mathbb{A} \,\}$$

Theorem 4 *Let* $h \in \mathbb{H}$*. Then*

$$h = \mathbf{P}_{\mathbb{H}}(x) \Leftrightarrow x - h \in (\mathbb{H} - h)^0 \quad \square$$

For a proof we refer to [6] or [8]. So apparently to find the metric projection, we need to calculate the dual cone of a translate of $\mathbb{H}$. Alternatively, the set $\mathbb{H}$ can be described in terms of the metric projections onto it - [6]:

$$\mathbb{H} = \{\, y \in \mathbb{X} \mid \mathbf{P}_{\mathbb{H}}(y) = y \,\} \tag{32}$$

But calculating $(\mathbb{H} - h)^0$ does not seem to be very tractable. So apart from seeking a convenient environment for $\mathbb{H}$, we must in addition find a more manageable structure for $\mathbb{H}$ in this environment.
This leads to the idea to decompose $\mathbb{H}$ in such a way that the different parts of this decomposition allow a relatively straightforward calculation of the metric projection onto them, and then hopefully we can re-combine the different metric projections to infer from them the metric projection onto $\mathbb{H}$. Clearly we should decompose $\mathbb{H}$ into convex parts. Recall that we noticed in section 2 when discussing the conical hull $con(\mathbb{H})$ of $\mathbb{H}$ that specifically *convex cones* are convenient objects in convex analysis. Now the 'source' of $\mathbb{H}$ is the set $\mathbb{Y}$, consisting of a *finite* number of elements. We have in this connection the following useful result:

Proposition 3

(1) The conical hull of $\mathbb{Y}$, $con(\mathbb{Y})$, *is a finitely generated cone, i.e.*

$$con(\mathbb{Y}) = \{\, y \in \mathbb{X} \mid \; y = \sum_{i=1}^{n} \tau_i y_i \,,\, y_i \in \mathbb{Y} \,,\, \tau_i \in \mathbb{R}^+ \,\}$$

(2) $y + con(\mathbb{Y})$ *is Chebyshev for any* $y \in \mathbb{S}$

Proof:

(1) see [6].

(2) If $con(\mathbb{Y})$ is Chebyshev, then so is its translate - see [6] - and so in view of the first part of theorem 3 we need to show that $con(\mathbb{Y})$ is closed. This is proved in [8] as one of the lemma's of J. Farkas in a $\mathbb{R}^n$-setting and, in particular because the $\{y_i\}$ are linearly independent, can be translated to our setting. In [6] a very nice proof is presented in our inner product space setting. Both references consider the situation where there may be linear dependence among the generators of the conical hull. $\square$

$con(\mathbb{Y})$ would be a very desirable 'building block' in our decomposition plan for $\mathbb{H}$. The following result shows that it is:

Proposition 4 *Let*

$$\begin{aligned} y_{min} &= \sum_{i=1}^{n} \alpha_i y_i \ , \ y_{max} = \sum_{i=1}^{n} \beta_i y_i \\ \mathbb{W} &= \{ w_i \in \mathbb{X} \mid w_i = -y_i \ , \ y_i \in \mathbb{Y} \} \end{aligned}$$

then

$$\begin{aligned} \mathbb{H} &= (y_{min} + con(\mathbb{Y})) \cap (y_{max} - con(\mathbb{Y})) \\ &= (y_{min} + con(\mathbb{Y})) \cap (y_{max} + con(\mathbb{W})) \end{aligned}$$

Proof:

$$\begin{aligned} \mathbb{H} &= \{ y \in \mathbb{X} \mid \ y = \sum_{i=1}^{n} \gamma_i y_i \ , \ \alpha_i \leq \gamma_i \leq \beta_i \} \\ &= \{ y \in \mathbb{X} \mid \ y = \sum_{i=1}^{n} \gamma_i y_i \ , \ \gamma_i \geq \alpha_i \} \cap \{ y \in \mathbb{X} \mid \ y = \sum_{i=1}^{n} \gamma_i y_i \ , \ \gamma_i \leq \beta_i \} \end{aligned}$$

$$\begin{aligned} \{ y \in \mathbb{X} \mid \ y = \sum_{i=1}^{n} \gamma_i y_i \ , \ \gamma_i \geq \alpha_i \} &= \\ \{ y \in \mathbb{X} \mid \ y = \sum_{i=1}^{n} (\gamma_i - \alpha_i) y_i + y_{min} \ , \ \gamma_i \geq \alpha_i \} &= \\ y_{min} + \{ y \in \mathbb{X} \mid \ y = \sum_{i=1}^{n} \tau_i y_i \ , \ \tau_i \geq 0 \} &= y_{min} + con(\mathbb{Y}) \end{aligned}$$

Likewise the other component of $\mathbb{H}$ $\square$

The decomposition of $\mathbb{H}$ is in two translated convex cones not unique. Indeed, there are $2^{(n-1)} - 1$ other similar decompositions. Because the translated cones in these decompositions play an important role in the sequel of this paper, we present them here as a corollary to proposition 4. But before we are in a position to do that, we need to introduce some additional mathematical objects. The following definition is taken from F. Deutsch [6]:

Definition 7 *$h \in \mathbb{H}$ is an extreme point of $\mathbb{H}$, if $f, g \in \mathbb{H}$, $0 < \lambda < 1$, and $h = \lambda f + (1-\lambda) g$ implies $f = g = h$.*

With reference to this definition, we introduce the *set of extreme points of* $\mathbb{H}$:

$$\mathbb{E} = \{r^1, \ldots, r^{2^n} \mid r^j = \sum_{i=1}^{n} \psi_i^j y_i \,;\, y_i \in \mathbb{Y}\,,\, \psi_i^j = \{\alpha_i \,\mathrm{or}\, \beta_i\}\,,\, j = 1, \ldots, 2^n\,, \\ \psi_i^j \neq \psi_i^k \,(j \neq k)\,\} \quad (33)$$

Apparently $y_{min}\,,\, y_{max} \in \mathbb{E}$. The pair of extreme points $\{y_{min}, y_{max}\}$ belongs to the *pairs of opposite extreme points* of $\mathbb{H}$ that we define next:

Definition 8 *$\{r^j, r^k\}\,(r^j\,,\, r^k \in \mathbb{E}\,,\, j \neq k\,)$ is a pair of opposite extreme points of $\mathbb{H}$ if $\psi_i^j \neq \psi_i^k\; \forall\, i$*

Without loss of generality we may assume that the set $\mathbb{E}$ is ordered in such a way that $\{\{r^{2j-1}, r^{2j}\} \mid j = 1, \ldots, 2^{n-1}\}$ are the opposite pairs of $\mathbb{E}$; in particular we assume that $\{r^1, r^2\} = \{y_{min}, y_{max}\}$.

Definition 9 *The translated convex cone associated with the extreme point $r^j \in \mathbb{E}$ is given by*

$$r^j + con(\mathbb{Z}^j) \quad (34)$$

where

$$\mathbb{Z}^j = \{z_1^j, \ldots, z_n^j \mid z_i^j = \begin{cases} y_i & if\, \psi_i^j = \alpha_i \\ w_i = -y_i & if\, \psi_i^j = \beta_i \end{cases} \} \quad (35)$$

Corollary 1 *Let $\{r^{2j-1}, r^{2j}\}\,(r^{2j-1}, r^{2j} \in \mathbb{E})$ be any pair of opposite extreme points of $\mathbb{H}$. Then*

$$\mathbb{H} = (r^{2j-1} + con(\mathbb{Z}^{2j-1})) \cap (r^{2j} + con(\mathbb{Z}^{2j})) \quad \square$$

So assuming the decomposition plan will work, in which we hope to infer the metric projection onto $\mathbb{H}$ from the metric projections onto its components, the problem we are facing next is calculating the metric projection onto a translated convex cone. This is taken up in the next section.

3 Explicit Calculation of the Metric Projection onto a Translated Convex Cone

We start with the following sharpening of theorem 4 for convex cones. For ease of presentation, and as we will substantiate at the end of this section, without loss of generality, we present or results mainly - with a few exceptions - for $r^1 + con(\mathbb{Z}^1) = y_{min} + con(\mathbb{Y})$.

Proposition 5 *Denote $(con(\mathbb{Y}) + y_{min})$ by $\mathbb{K}^1$ and let $y_0 \in \mathbb{K}^1$. Then*

$$y_0 = \mathbf{P}_{\mathbb{K}^1}(x) \Leftrightarrow \langle x - y_0, y\rangle \leq 0\, \forall\, y \in con(\mathbb{Y}) \;\;\&\;\; \langle x - y_0, y_0 - y_{min}\rangle = 0$$

Proof: $y_0 = \mathbf{P}_{\mathbb{K}^1}(x) \Leftrightarrow (x - y_0) \in (\mathbb{K}^1 - y_0)^0 \Leftrightarrow \langle x - y_0, y - (y_0 - y_{min})\rangle \leq 0 \,\forall\, y \in con(\mathbb{Y})$. The non-positivity condition on the inner product holds for all $y \in con(\mathbb{Y})$, and so in particular for the following two choices:

$$\begin{aligned} y = 0 \quad &\Rightarrow \quad \langle x - y_0, y_0 - y_{min}\rangle \geq 0 \\ y = 2(y_0 - y_{min}) \quad &\Rightarrow \quad \langle x - y_0, y_0 - y_{min}\rangle \leq 0 \quad \square \end{aligned}$$

The proof given here is a straightforward adaptation to the proof for the 'untranslated' case which may be found in [6] and in [8].
There is a useful corollary to proposition 5. Before stating this result, we have to introduce the following sets:

Definition 10 *The translated polar cone of* $con(\mathbb{Z}^j) + r^j$ *(cf. definition 9), denoted by* $(con(\mathbb{Z}^j) + r^j)_0$*, is the following convex set:*

$$(con(\mathbb{Z}^j) + r^j)_0 = \{c \in \mathbb{X} \mid \langle (c - r^j), z\rangle \leq 0 \,\forall\, z \in con(\mathbb{Z}^j)\,\} \tag{36}$$

The usefulness of the translated polar cones will become clear when we view them in perspective with respect to the dual cones of $con(\mathbb{Z}^j) + r^j$, and of $\mathbb{H}$ at the end of this section. For now we restrict ourselves to a corollary to proposition 5 where they play a major role.

Corollary 2 *Let* $\mathbb{K}^j = con(\mathbb{Z}^j) + r^j$ *and let* $x \in \mathbb{K}^j_0$*, then* $\mathbf{P}_{\mathbb{K}^j}(x) = r^j$*.*

Proof: It is sufficient to restrict ourselves to the case $j = 1$. We note that, by substituting y_{min} for y_0 in proposition 5 both conditions are satisfied. $\square$

For the situation where $x \notin (con((Y) + y_{min})_0$, we have, according to proposition 5, reduced the problem of finding the metric projection of x onto $con(\mathbb{Y}) + y_{min}$ to checking the non-positivity of the inner product between $x - y_0$ and all elements of $con(\mathbb{Y})$, and the orthogonality between $x - y_0$ and $y_0 - y_{min}$. For this to be possible at all, we need a representation of all elements involved in terms of, or in any case related to, the elements y_i of $\mathbb{Y}$. But what has $x \in \mathbb{X}$ got to do with $\mathbb{Y}$. The following result, the proof of which may again be found in F. Deutsch [6] gives a very nice way out of this dilemma:

Theorem 5 The Reduction Principle

$$\mathbf{P}_{\mathbb{K}^1}(\mathbf{P}_{\mathbb{S}}(x)) = \mathbf{P}_{\mathbb{K}^1}(x) = \mathbf{P}_{\mathbb{S}}(\mathbf{P}_{\mathbb{K}^1}(x)) \quad \square$$

The first equality in Theorem 5 is of importance for us: we establish the metric projection of x onto $\mathbb{S}$ first - there is only one, since by Theorem 3 $\mathbb{S}$ is Chebyshev - and subsequently use $\mathbf{P}_{\mathbb{S}}(x)$ instead of x in Proposition 5.
To find the best approximation to x from $\mathbb{S}$, we introduce the following set: Let

$$\mathbb{B} = \{\, b_i \in \mathbb{X} \mid \; i = 1, \ldots, n\,\} \tag{37}$$

be an orthonormal basis for $\mathbb{S}$, i.e. $\mathbb{S} = span(\mathbb{Y}) = span(\mathbb{B})$, and $\langle b_i, b_j\rangle = 1$ if $i = j$ and $\langle b_i, b_j\rangle = 0$ otherwise. In particular $\mathbb{B}$ can be constructed from $\mathbb{Y}$ through the *Gram-Schmidt orthogonalization process* - see e.g. [6]. The best approximation to x from $\mathbb{S}$ is simply the Fourier series expansion of $\mathbf{P}_{\mathbb{S}}(x)$ relative to $\mathbb{B}$:

$$\mathbf{P}_{\mathbb{S}}(x) := s = \sum_{i=1}^{n} \langle b_i, x\rangle b_i \tag{38}$$

But to apply proposition 5 we would like to have $\mathbf{P}_{\mathbb{S}}(x)$ directly in terms of the elements y_i of $\mathbb{Y}$. This turns out to be an aspect that turns up quite frequently: we commute in $\mathbb{S}$ between the orthonormal basis $\mathbb{B}$ in which we can calculate easily using the Fourier-coefficients as coordinates, and the 'given' basis $\mathbb{Y}$ in which we can interpret our results using the barycentric coefficients as coordinates. Writing the expansion of $\mathbf{P}_{\mathbb{S}}(x)$ with respect to the basis $\mathbb{Y}$ as

$$\mathbf{P}_{\mathbb{S}}(x) = \sum_{i=1}^{n} \delta_i y_i \quad (\, \delta_i \in \mathbb{R}\,) \tag{39}$$

what we are looking for is the relation between $\{\delta_1, \ldots, \delta_n\}$ and $\{\langle b_1, x\rangle, \ldots, \langle b_n, x\rangle\}$. This question is addressed by P. Halmos [7] and we follow his development:

Consider the linear transformation $\mathbf{T} : \mathbb{S} \to \mathbb{S}$ defined by $\mathbf{T}y_i = b_i$. The matrix of this transformation with respect to the basis $\mathbb{Y}$ is denoted by (t_{ij}), i.e. $b_j = \mathbf{T}y_j = \sum_{i=1}^{n} t_{ij} y_i$ - concerning this notation that is not conform the 'usual' matrix-vector notation in $\mathbb{R}^n$, we cite Paul Halmos [7] that this fact is"a perversity not of the author, but of nature". It follows that:

$$\begin{aligned} (t_{ij}) &= \mathbf{G}^{-1}(y_1, y_2, \ldots, y_n)(u_{ij}) \\ (u_{ij}) &= (\langle y_i, b_j\rangle) \end{aligned} \tag{40}$$

where $\mathbf{G}(y_1, y_2, \ldots, y_n)$ is the Gram matrix defined in equation (15). Recall that $\mathbf{G}(y_1, y_2, \ldots, y_n)$ - [6] - and (t_{ij}) - [7] - are invertible, and so *a fortoriori* (u_{ij}) is invertible. The relation between the sets of coordinates of s in respectively equation (39) and (38) follows from a direct computation:

$$\delta_i = \sum_{j=1}^{n} t_{ij} \langle b_j, x\rangle \tag{41}$$

Note that equation (41) is equivalent to solving the normal equations in the δ_i - see equation (14):

$$\sum_{i=1}^{n} \delta_i \langle y_i, y_j\rangle = \langle s, y_j\rangle \quad (j = 1, \ldots, n)$$

Likewise using the inverse transformation $\mathbf{V} : \mathbb{S} \rightarrow \mathbb{S}$ defined by $y_j = \mathbf{V} b_j$ and with the matrix of this transformation with respect to $\mathbb{B}$ denoted by (v_{ij}), i.e. $y_j = \mathbf{V} b_j = \sum_{i=1}^{n} v_{ij} b_i$ where

$$(v_{ij}) = (u_{ij})^{\,t} \tag{42}$$

where the superscript t denotes *transposition*, and so the Fourier coefficients of s in equation (38) as function of the barycentric coefficients in equation (39) is given by

$$\langle b_i, x \rangle = \sum_{j=1}^{n} v_{ij} \delta_j \tag{43}$$

Note that

$$(u_{ij})(v_{ij}) = \mathbf{G}(y_1, y_2, \ldots, y_n)$$

Let us now return to proposition 5. In view of equations (32) we assume that $s \notin \mathbb{H}$:

$$s \notin \mathbb{H} \Leftrightarrow \delta \notin \Gamma \tag{44}$$

Moreover, in view of corollary 2 we assume that $s \notin (con(\mathbb{Y}) + y_{min})_0$, and $s \notin (con(\mathbb{W}) + y_{max})_0$; we present the conditions for this being the case at the end of this section.
We concentrate first on the orthogonality condition in proposition 5. The elements involved in this condition have the following representations with respect to the basis $\mathbb{Y}$ of $\mathbb{S}$:

$$\begin{aligned} y_0 &:= y_{min} + y = \sum_{i=1}^{n} (\alpha_i + \rho_{0_i}) y_i \qquad (45) \\ s - y_0 &= \sum_{i=1}^{n} (\delta_i - \rho_{0_i} - \alpha_i) y_i := \sum_{i=1}^{n} \phi_i y_i \\ y_0 - y_{min} &= \sum_{i=1}^{n} \rho_{0_i} y_i \end{aligned}$$

Using the base transformation $\mathbf{V} : \mathbb{S} \rightarrow \mathbb{S}$ we have the following equivalent representations for $s - y_0$ and $y_0 - y_{min}$ with respect to the basis $\mathbb{B}$ of $\mathbb{S}$:

$$\begin{aligned} s - y_0 &= \sum_{i=1}^{n} \langle b_i, s - y_0 \rangle b_i \quad \Big(\langle b_i, s - y_0 \rangle = \sum_{j=1}^{n} v_{ij} \phi_j \Big) \qquad (46) \\ y_0 - y_{min} &= \sum_{i=1}^{n} \langle b_i, y_0 - y_{min} \rangle b_i \quad \Big(\langle b_i, y_0 - y_{min} \rangle = \sum_{j=1}^{n} v_{ij} \rho_{0_j} \Big) \end{aligned}$$

$s-y_0$ and y_0-y_{min} are orthogonal to each other if $s-y_0$ is in the subspace of $\mathbb{S}$ that is the orthogonal complement of the subspace of $\mathbb{S}$ of which y_0-y_{min} is an element. In other words, the Fourier-coefficients of the representation of $s-y_0$ with respect to $\mathbb{B}$ that may be non-zero, *must* be zero in the representation of y_0-y_{min} with respect to $\mathbb{B}$, and the other way around.
To see how this works out let us take a 'candidate solution element' y_c from $con(\mathbb{Y})+y_{min}$ with representation

$$y_c = \sum_{i=1}^{n} \rho_{c_i} y_i + y_{min} \tag{47}$$

and to be specific let us assume that $y_c - y_{min} \in span\{\{b_1,\ldots,b_m\}\}$ & $s-y_c \in span\{\{b_{m+1},\ldots,b_n\}\}$ or equivalently $y_c-y_{min} \notin span\{\{b_{m+1},\ldots,b_n\}\}$ & $s-y_c \notin span\{\{b_1,\ldots,b_m\}\}$, and, using equations (45) and (46) with y_c substituted for y_0 this is in turn equivalent to the following system of linear equations in the unknown barycentric coefficients $\rho_c = (\rho_{c_i})$ of y_c:

$$(v_{ij})\rho_c = \begin{pmatrix} \langle b_1, s-y_{min}\rangle \\ \cdots \\ \cdots \\ \langle b_m, s-y_{min}\rangle \\ 0 \\ . \\ 0 \end{pmatrix} \tag{48}$$

where the $n \times n$ matrix (v_{ij}) is defined in equation (42). Hence there is a unique solution ρ_c - recall that (v_{ij}), being the matrix of the base transformation $\mathbf{V}: \mathbb{S} \to \mathbb{S}$, is non-singular. Indeed,

$$\rho_c = (t_{ij}) \begin{pmatrix} \langle b_1, s-y_{min}\rangle \\ \cdots \\ \cdots \\ \langle b_m, s-y_{min}\rangle \\ 0 \\ . \\ 0 \end{pmatrix} \tag{49}$$

with (t_{ij}) given in equation (40). Of course we must check that y_c really is in $con(\mathbb{Y})+y_{min}$:

$$y_c \in con(\mathbb{Y}) + y_{min} \Leftrightarrow \rho_c \geq 0 \tag{50}$$

We note that inequalities for vectors are to be understood component wise. Suppose y_c is in $con(\mathbb{Y})+y_{min}$; our candidate solution must pass subsequently the non-positivity test from proposition 5 to qualify itself finally as a solution:

$$\begin{aligned} \langle s-y_c, y\rangle \le 0\,\forall\, y \in con(\mathbb{Y}) &\quad \Leftrightarrow \\ \langle s-y_c, y_i\rangle \le 0\,\forall\, i \in \{1,\dots,n\} &\quad \Leftrightarrow \\ \mathbf{G}(y_1,\dots,y_n)(\delta-\rho_c-\alpha) \le 0 & \end{aligned} \tag{51}$$

We would like to add here that the test (50) is redundant in the sense that a candidate solution satisfying the orthogonality condition of proposition 5 that passes the test (51) is an element of $con(\mathbb{Y})+y_{min}$. The point is that we need ρ_c anyway for the test (51), and so if a candidate solution satisfying the orthogonality condition does *not* pass the test (50), we do not have to bother about the test (51), and instead immediately try a next candidate solution.

And so finally, realizing that there are $\sum_{i=1}^{n-1}\binom{n}{i} = 2^n-2$ ways, with $\binom{\cdot}{\cdot}$ the binomial coefficient, in which we can split $\mathbb{S}$ into orthogonal complements, we can state our main result about the explicit calculation of the metric projection onto $con(\mathbb{Y})+y_{min}$. However, before doing so, we need to introduce the following finite subset of $\mathbb{R}^n$:

$$\Xi = \overbrace{\left\{ \begin{pmatrix} \langle b_1, s-y_{min}\rangle \\ \cdots \\ \cdots \\ \langle b_{n-1}, s-y_{min}\rangle \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ \langle b_n, s-y_{min}\rangle \end{pmatrix} \right\}}^{\sum_{i=1}^{n-1}\binom{n}{i}\ elements} \tag{52}$$

where the vectors of this set are composed from all possible choices of elements from the two $\mathbb{R}^n$ vectors $(\langle s-y_{min}, b_i\rangle)$ and 0 with *at least one element from each.*

Theorem 6 *Suppose* $\mathbf{P}_{\mathbb{S}}(x) \notin \mathbb{H}$ & $\mathbf{P}_{\mathbb{S}}(x) \notin (con(\mathbb{Y})+y_{min})_0$. *There exists a unique element* $\xi_0 \in \Xi$ *such that*

$$\mathbf{P}_{con(\mathbb{Y})+y_{min}}(x) = \sum_{i=1}^{n} \rho_{0_i} y_i + y_{min}$$

where $\rho_0\,(=(\rho_{0_i}))$ *is given by*

$$\rho_0 = (t_{ij})\xi_0$$

with (t_{ij}) *the matrix of the base transformation* $\mathbf{T}:\mathbb{S}\to\mathbb{S}$ *given in equation* (40)*. In particular the best approximation to* $x \in \mathbb{X}$ *from* $con(\mathbb{Y})+y_{min}$ *is found in at most* (2^n-2) *steps.*

Proof: In view of our detailed analysis presented in this section, we can restrict ourselves to the claim that the result is obtained in *at most* (2^n-2) steps: the existence and uniqueness of the solution is guaranteed by the second part of proposition 3, so as soon as we have found a 'candidate solution' y_c by picking an arbitrary element from the set Ξ that passes the tests (50) and (51) we can stop. $\Box$

Assuming that the inner products can be calculated relatively easily, the different 'steps' are very straightforward and not at all computationally intensive.

Clearly the results of this section hold, *mutadis mutandis*, for the other translated convex cones $r^j + con(\mathbb{Z}^j)$ $(j = 2, \ldots, 2^n)$. So in particular they hold for the translated convex cone associated with the opposite extreme point of y_{min}, $con(\mathbb{W}) + y_{max} = y_{max} - con(\mathbb{Y})$ which, according to proposition 4 and corollary 1, gives together with $y_{min} + con(\mathbb{Y})$ a decomposition of $\mathbb{H}$. In fact, representing the candidate solution for the metric projection onto $y_{max}+con(\mathbb{W})$ by

$$y_c = y_{max} + \sum_{i=1}^{n} \rho_{c_i} w_i = \sum_{i=1}^{n} (\beta_i y_i + \rho_{c_i} w_i) = \sum_{i=1}^{n} (\beta_i - \rho_{c_i}) y_i \tag{53}$$

the most important changes are that equations (49), (50), and (51) should be replaced by, respectively

$$\rho_c = -(t_{ij}) \begin{pmatrix} \langle b_1, s - y_{max} \rangle \\ \cdots \\ \cdots \\ \langle b_m, s - y_{max} \rangle \\ 0 \\ . \\ 0 \end{pmatrix} \tag{54}$$

$$y_c \in y_{max} - con(\mathbb{Y}) \Leftrightarrow \rho_c \geq 0 \tag{55}$$

$$\langle s - y_c, w \rangle \leq 0 \, \forall \, w \in con(\mathbb{W}) \Leftrightarrow \mathbf{G}(y_1, \ldots, y_n)(\beta - \rho_c - \delta) \leq 0 \tag{56}$$

Before closing this section, we come back to the translated polar cones introduced in definition 10. First of all we note in view of the Reduction Principle (theorem 5) the only relevant part of these dual - and translated polar cones is their intersection with $\mathbb{S}$. For that reason they are to be understood as a subsets of $\mathbb{S}$, leaving their notation unchanged. In view of proposition 4 and corollary 1, we always have $H^0 \supset (con(\mathbb{Z}^{2j-1}) + r^{2j-1})^0 + (con(\mathbb{Z}^{2j}) + r^{2j})^0$ - see F.Deutsch [6]. The following theorem collects a number of relations for the dual - and translated polar cones. In particular we present a description of $\mathbb{H}^0$ in terms of its metric projection onto $\mathbb{H}$ using the translated polar cones. Before stating the theorem, we need to introduce some additional objects related to the extreme points.

Definition 11 *Let $r^j \in \mathbb{E}$ - cf.* (33) *- with barycentric coefficients $\{\psi_1^j, \ldots, \psi_n^j\}$.*
$r^k \in \mathbb{E}$ *is an adjacent extreme point of* r^j *if* $\begin{cases} \psi_i^k \neq \psi_i^j \text{ for exactly one } i \in \{1, \ldots, n\} \\ \psi_i^k = \psi_i^j \text{ for all other } i \in \{1, \ldots, n\} \end{cases}$
The collection of n adjacent extreme points of $r^j \in \mathbb{E}$ is denoted by $\mathbb{E}_{r^j}$.

The extremal subset of $\mathbb{H}$ associated with $r^j \in \mathbb{E}$ and one of its adjacent extreme points $r^{j_i} \in \mathbb{E}_{r^j}$ is given by

$$\mathbb{E}_{r^j}^{r^{j_i}} = \{h \in \mathbb{H} \mid h = \lambda r^j + (1-\lambda) r^{j_i} \, , \, 0 \leq \lambda \leq 1\}$$

Theorem 7 *Let $j \in \{1, \ldots, 2^n\}$.*

(1)
$$(r^j + con(\mathbb{Z}^j))^0 = (r^j + con(\mathbb{Z}^j))_0 \Leftrightarrow r^j = 0$$

(2)
$$(r^j + con(\mathbb{Z}^j))_0 \neq \emptyset$$

(3) Let $\mathbb{N}_\epsilon$ be any ϵ-environment of $0 \in \mathbb{S}$. Then

$$\mathbb{N}_\epsilon \subset \mathbb{H} \Rightarrow \mathbb{H}^0 = \emptyset \Rightarrow (r^j + con(\mathbb{Z}^j))^0 = \emptyset$$

(4)
$$\mathbb{N}_\epsilon \subset (r^j + con(\mathbb{Z}^j)) \Rightarrow (r^j + con(\mathbb{Z}^j))^0 = \emptyset$$

(5) Assume $\mathbb{N}_\epsilon \not\subset \mathbb{H}$, and let $r^k \in \mathbb{E}$ be such that $\left\|r^k\right\| < \left\|r^j\right\| \forall \, r^j \in \mathbb{E}$, $(j \neq k)$. Consider the set

$$\mathbb{D} = \mathbb{H}^0 \setminus (\bigcup_{i=1}^{n} ((r^{k_i} + con(\mathbb{Z}^{k_i}))_0 \mid r^{k_i} \in \mathbb{E}_{r^k}) \cup (r^k + con(\mathbb{Z}^k))_0)$$

$$s \in \mathbb{D} \Rightarrow \mathbf{P}_{\mathbb{H}}(s) \in (\mathbb{E}_{r^k}^{r^{k_{i_1}}} + \cdots + \mathbb{E}_{r^k}^{r^{k_{i_{n-1}}}}) \setminus ((\bigcup_{l=1}^{n-1} \{r^{k_{i_l}}\}) \cup \{r^k\})$$

where $r^{k_{i_l}} \in \mathbb{E}_{r^k}$ $(l = 1, \ldots, n-1)$ is one of the n possible choices of $(n-1)$ adjacent extreme points of r^k

(6)
$$s \in \mathbb{H}^0 \setminus \mathbb{D} \Rightarrow \mathbf{P}_{\mathbb{H}}(s) \begin{cases} = r^k \text{ if } s \in (r^k + con(\mathbb{Z}^k))_0 \\ \in \mathbb{E}_{r^k} \text{ otherwise} \end{cases}$$

Proof:

(5) The set $\mathbb{D}$ is *disconnected* - see J. Lee [9] - with convex components. If s is in one of these components it follows from our construction that the point of $\mathbb{H}$ closest to s is an element of the sum of extremal subsets of $\mathbb{H}$, but without the extreme points, it faces.

(6) This follows from corollary 2. $\square$

Let us see what we have got thus far. We have seen in corollary 2 that for $s \in (con(\mathbb{Z}^j) + r^j)_0$ $(j \in \{1, \ldots, 2^n\})$, where

$$s \in (con(\mathbb{Z}^j) + r^j)_0 \Leftrightarrow \mathbf{F}(y_1, \ldots, y_n, z_1^j, \ldots, z_n^j)(\delta - \psi^j) \leq 0 \qquad (57)$$
$$(\mathbf{F}(y_1, \ldots, y_n, z_1^j, \ldots, z_n^j))_{i,k} = \langle y_i, z_k^j \rangle \; (i\,,\, k = 1, \ldots, n)$$

the metric projection of s onto the translated cones is straightforward. Theorem 7 presents results for finding $\mathbf{P}_{\mathbb{H}}(s)$ with $s \in \mathbb{H}^0$, where

$$s \in \mathbb{H}^0 \Leftrightarrow \mathbf{G}(y_1, \ldots, y_n)\delta \leq -\langle s, y_{min} \rangle \qquad (58)$$

But establishing that the metric projection onto $\mathbb{H}$ is an element of a sum of extremal sets of $\mathbb{H}$ is not nearly good enough when the goal is to *compute* this metric projection. And on top of that, what to do when
$s \in \mathbb{S} \setminus ((\bigcup_{j=1}^{2^n}(r^j + con(\mathbb{Z}^j))_0) \cup \mathbb{H})$; this situation is not covered at all at this stage. In other words, it is high time we concentrate on the next step in our program, viz. inferring the metric projection onto $\mathbb{H}$ from the metric projections onto the members of its decomposition. This is addressed in the next, final section.

4 The Boyle - Dykstra Theorem

The Boyle-Dykstra theorem establishes the convergence of an iterative procedure that computes best approximations from an intersection $\cap_{i=1}^n \mathbb{C}_i$ of a finite number of closed convex subsets $\mathbb{C}_i$ of a *Hilbert* space from the best approximations of the individual sets $\mathbb{C}_i$. - A comprehensive treatment of this theorem can be found in the recent book of F. Deutsch [6]. This is exactly our situation, because thanks to the Reduction Principle theorem 5, proposition 4, and corollary 1 we are looking for the best approximation from $(con(\mathbb{Z}^{2j-1}) + r^{2j-1}) \cap (con(\mathbb{Z}^{2j}) + r^{2j})$ $(j \in \{1, \ldots, 2^{n-1}\})$ to $\mathbf{P}_{\mathbb{S}}(x)$, and according to proposition 3 the translated cones are closed, convex subsets of the *Hilbert* (sub-)space $\mathbb{S}$.
Now write

$$\mathbb{K}^j = con(\mathbb{Z}^j) + r^j$$

and let, for $n \in \mathbb{N}$, $[n]$ denote 'n modulo 2', i.e.

$$[n] := \{1, 2\} \cap \{n - 2k \mid k = 0, 1, 2, \ldots\}$$

Without loss of generality we present the Boyle-Dykstra theorem for the translated cones $\mathbb{K}^1$ and $\mathbb{K}^2$ associated with the pair of opposite extreme points y_{min} and y_{max} see definitions 8 and 9, and corollary 1.

Theorem 8 The Boyle-Dykstra theorem

Let $\mathbf{P}_{\mathbb{S}}(x)$ be the best approximation to $x \in \mathbb{X}$ from $\mathbb{S}$, and let $\mathbf{P}_{\mathbb{S}}(x) \notin \mathbb{H}$. Construct the following sequence $\{x_n\}$ in $\mathbb{S}$:

$$\begin{aligned} x_0 &= \mathbf{P}_{\mathbb{S}}(x), \quad e_{-1} = e_0 = 0, \\ x_n &= \mathbf{P}_{\mathbb{K}^{[n]}}(x_{n-1} + e_{n-2}), \quad (\textit{using theorem 6}) \\ e_n &= x_{n-1} + e_{n-2} - x_n \quad (n = 1, 2, \ldots) \end{aligned}$$

This sequence converges to the best approximation from $\mathbb{H}$ in the following way:

$$\lim_{n\to\infty} \|x_n - \mathbf{P}_{\mathbb{H}}(x)\| = 0$$

with $\|\cdot\|$ the norm on $\mathbb{X}$ $\square$

For a proof of the Boyle-Dykstra theorem we refer to the excellent recent book of F. Deutsch [6], in which also references to different applications of this theorem may be found.

We are now ready to present our final result.

Theorem 9 *Let $\mathbf{P}_{\mathbb{S}}(x)$ be the best approximation to $x \in \mathbb{X}$ from $\mathbb{S}$. Then the best approximation to x from $\mathbb{H}$, $\mathbf{P}_{\mathbb{H}}(x)$, is given by one of the three following cases:*

(1) If $\mathbf{P}_{\mathbb{S}}(x) \in \mathbb{H}$ (cf. (44)), then $\mathbf{P}_{\mathbb{H}}(x) = \mathbf{P}_{\mathbb{S}}(x)$

(2) If $\mathbf{P}_{\mathbb{S}}(x) \in \mathbb{K}_0^j$ ($j \in \{1, \ldots, 2^n\}$) (cf. (57)), then $\mathbf{P}_{\mathbb{H}}(x) = r^j$

(3) If $\mathbf{P}_{\mathbb{S}}(x) \in \mathbb{S} \setminus ((\bigcup_{j=1}^{2^n} \mathbb{K}_0^j) \cup \mathbb{H})$, then $\mathbf{P}_{\mathbb{H}}(x) = \mathbf{P}_{\mathbb{K}^j}(\mathbf{P}_{\mathbb{S}}(x))$ where r^j is such that $\left\|\mathbf{P}_{\mathbb{S}}(x) - r^j\right\| \leq \left\|\mathbf{P}_{\mathbb{S}}(x) - r^i\right\| \; \forall\, r^i \in \mathbb{E}$, $(r^i \neq r^j)$

Proof:

(1) See equation (32).

(2) $\mathbf{P}_{\mathbb{S}}(x) \in \mathbb{K}_0^j \Rightarrow \mathbf{P}_{\mathbb{K}^j}(\mathbf{P}_{\mathbb{S}}(x)) = r^j$. Without loss of generality we may assume that j is odd, i.e. $j = 2k-1$ for some $k \in \{1, \ldots, 2^{n-1}\}$. But $r^j = r^{2k-1} \in \mathbb{K}^{2k}$, and $\mathbb{H} = \mathbb{K}^{2k-1} \cap \mathbb{K}^{2k}$. The result now follows from the Boyle-Dykstra theorem, because the sequence $\{x_n\}$ constructed in the theorem converges immediately to the constant sequence $\{r^j\}$.

(3) $\mathbf{P}_{\mathbb{S}}(x)$ is in one of the components of the disconnected set $\mathbb{S} \setminus ((\bigcup_{j=1}^{2^n} \mathbb{K}_0^j) \cup \mathbb{H})$. In fact it follows from our construction that $\mathbf{P}_{\mathbb{S}}(x)$ is in the component that

looks out at the subset $(\mathbb{E}_{r^j}^{r^{j_{i_1}}} + \cdots + \mathbb{E}_{r^j}^{r^{j_{i_{n-1}}}}) \setminus ((\cup_{l=1}^{n-1}\{r^{j_{i_l}}\}) \cup \{r^j\})$ of $\mathbb{K}^j$, where r^j is such that $\left\|\mathbf{P}_{\mathbb{S}}(x) - r^j\right\| \leq \left\|\mathbf{P}_{\mathbb{S}}(x) - r^i\right\| \forall\, r^i \in \mathbb{E}$, $(r^i \neq r^j)$ and $r^{j_{i_l}} \in \mathbb{E}_{r^j}$ $(l = 1, \ldots, n-1)$ is one of the n possible choices of $(n-1)$ adjacent extreme points of r^j. Without loss of generality we may again assume that j is odd, i.e. $j = 2k - 1$ for some $k \in \{1, \ldots, 2^{(n-1)}\}$. Using theorem 6 we compute $\mathbf{P}_{\mathbb{K}^{2k-1}}(\mathbf{P}_{\mathbb{S}}(x)) \in (\mathbb{E}_{r^j}^{r^{j_{i_1}}} + \cdots + \mathbb{E}_{r^j}^{r^{j_{i_{n-1}}}}) \setminus ((\cup_{l=1}^{n-1}\{r^{j_{i_l}}\}) \cup \{r^j\}) \subset \mathbb{K}^{2k-1}$. But $\mathbf{P}_{\mathbb{K}^{2k-1}}(\mathbf{P}_{\mathbb{S}}(x)) \in \mathbb{K}^{2k}$. The result follows again from the Boyle-Dykstra theorem. $\square$

Note that theorem 9 and theorem 7 combined gives a complete description of $\mathbb{H}^0$ in terms of its metric projection onto $\mathbb{H}$.
In [10] the results of this paper are applied to an important problem in oil industry.

References

[1] Dan Amir, *Characterizations of Inner Product Spaces,* Birkäuser Verlag, Basel, 1986.

[2] Alexander Barvinok, *A Course in Convexity,* American Mathematical Society, Providence, Rhode Island, 2002.

[3] Adi Ben-Israel and Thomas N.E. Greville, *Generalized Inverses,* Springer Verlag, New York, second edition, 2003.

[4] Åke Björck, *Numerical Methods for Least Squares Problems,* SIAM, Philadelphia, 1996.

[5] Ole Christensen, *An Introduction to Frames and Riesz Bases,* Birkäuser Verlag, Boston, 2003

[6] Frank Deutsch, *Best Approximation in Inner Product Spaces,* Springer Verlag, New York, 2001.

[7] Paul R. Halmos, *Finite-Dimensional Vector Spaces,* Springer Verlag, New York, Reprinted second edition, 1987.

[8] Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal, *Fundamentals of Convex Analysis,* Springer Verlag, Berlin, 2001

[9] John M. Lee, *Introduction to Topological Manifolds,* Springer Verlag, New York, 2000.

[10] Hennie Poulisse, *Reconciliation reconciled,* Shell Research Report, *in preparation*, Rijswijk, 2004.

[11] Roger Webster, *Convexity,* Oxford University Press, Reprinted edition, 2002.