Computing Approximate Border Bases

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An overview article

From Oil Fields to Hilbert Schemes

will appear in the proceedings of this conference.

Contents

1 – Border Bases

It is difficult to get a man to understand something when his salary depends upon his not understanding it.

(Upton Sinclair)

K field

$P = K[x_1, \dots, x_n]$ polynomial ring over K

$I \subseteq P$ zero-dimensional polynomial ideal (i.e. $\dim_K(P/I) < \infty$)

$\mathbb{T}^n = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_i \geq 0\}$ monoid of terms

Definition of Border Bases

Definition 1.1 **(a)** A (finite) set $\mathcal{O} \subset \mathbb{T}^n$ is called an **order ideal** if $t \in \mathcal{O}$ and $t' \mid t$ implies $t' \in \mathcal{O}$.

(b) Let $\mathcal{O}$ be an order ideal. The set $\partial\mathcal{O} = (x_1\mathcal{O} \cup \cdots \cup x_n\mathcal{O}) \setminus \mathcal{O}$ is called the **border** of $\mathcal{O}$.

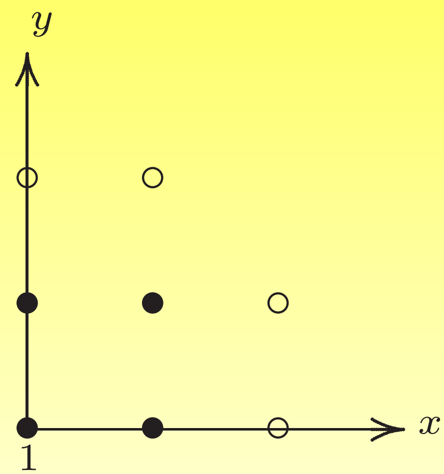
(c) Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal and $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ its border. A set of polynomials $\{g_1, \ldots, g_\nu\} \subset I$ of the form

$$g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$$

with $c_{ij} \in K$ and $t_i \in \mathcal{O}$ is called an **$\mathcal{O}$-border prebasis** of I.

(d) An $\mathcal{O}$-border prebasis of I is called an **$\mathcal{O}$-border basis** of I if the residue classes of the terms in $\mathcal{O}$ are a K-vector space basis of P/I.

A Picture of an Order Ideal and its Border



Example 1.2 Given a term ordering σ, the ideal I has a border basis with respect to $\mathcal{O}_\sigma(I) = \mathbb{T}^n \setminus \mathrm{LT}_\sigma(I)$, namely the one given by $G = \{g_1, \ldots, g_\nu\}$ with

$$g_j = b_j - \mathrm{NF}_{\sigma,I}(g_j)$$

Example 1.3 The ideal $I = \langle x^2 + xy + y^2,\ x^3,\ x^2y,\ xy^2,\ y^3\rangle$ in $\mathbb{R}[x, y]$ has a border basis with respect to $\mathcal{O} = \{1,\ x,\ y,\ x^2,\ y^2\}$. But this order ideal is not of the form $\mathcal{O}_\sigma(I)$ because $\mathrm{LT}_\sigma(x^2 + xy + y^2) \in \{x^2, y^2\}$.

2 – Properties of Border Bases

The list of the theorems I knew
made limericks end at line two.
(Anonymous)

In the following, we use the following notation:

$\mathcal{O} = \{t_1, \dots, t_\mu\}$ order ideal

$\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$ border of $\mathcal{O}$

$G = \{g_1, \dots, g_\nu\}$ is an $\mathcal{O}$-border prebasis, where

$g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$ with $c_{ij} \in K$

$I = \langle g_1, \dots, g_\nu \rangle$ ideal generated by G

Proposition 2.1 **(Existence of Border Bases)**

(a) *The ideal I need not have an $\mathcal{O}$-border basis. But if it does, the $\mathcal{O}$-border basis of I is uniquely determined.*

(b) *If $\mathcal{O}$ is of the form $\mathbb{T}^n \setminus \mathrm{LT}_\sigma(I)$ for some term ordering σ, then I has an $\mathcal{O}$-border basis. It contains the reduced σ-Gröbner basis of I.*

Proposition 2.2 *There exists a* **Division Algorithm** *for border prebases.*

Proposition 2.3 *The rewriting system defined by the rules $b_j \longrightarrow \sum_{i=1}^{\mu} c_{ij} t_i$ is confluent. (But it is in general not* **terminating**, *i.e. not Noetherian.)*

Characterization Using Multiplication Matrices

For $r \in \{1, \ldots, n\}$, we define the r-th **formal multiplication matrix** $\mathcal{A}_r$ as follows:

Multiply $t_i \in \mathcal{O}$ by x_r. If $x_r t_i = b_j$ is in the border of $\mathcal{O}$, rewrite it using the prebasis polynomial $g_j = b_j - \sum_{k=1}^{\mu} c_{kj} t_k$ and put $(c_{1j}, \ldots, c_{\mu j})$ into the i-th column of $\mathcal{A}_r$. But if $x_r t_i = t_j$ then put the j-th unit vector into the i-th column of $\mathcal{A}_r$.

Theorem 2.4 *(Mourrain)*
The set G is the $\mathcal{O}$-border basis of I if and only if the formal multiplication matrices commute, i.e. iff

$$\mathcal{A}_i \, \mathcal{A}_j = \mathcal{A}_j \, \mathcal{A}_i \qquad \textit{for } 1 \le i < j \le n.$$

3 – Neighbors

Under capitalism, man exploits man.
Under communism, it's just the opposite.
(John Kenneth Galbraith)

Definition 3.1 Let $b_i, b_j \in \partial\mathcal{O}$ be two distinct border terms.

(a) The border terms b_i and b_j are called **next-door neighbors** if $b_i = x_k\, b_j$ for some $k \in \{1, \ldots, n\}$.

(b) The border terms b_i and b_j are called **across-the-street neighbors** if $x_k\, b_i = x_\ell\, b_j$ for some $k, \ell \in \{1, \ldots, n\}$.

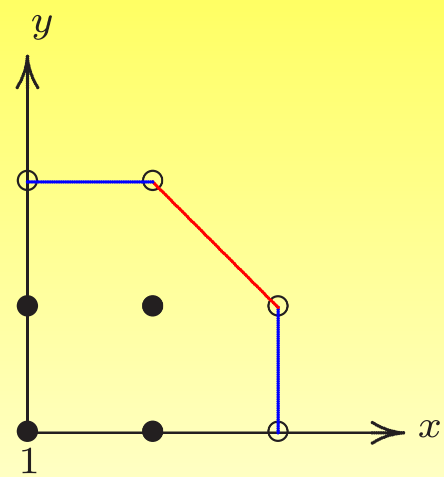
(c) The border terms b_i and b_j are called **neighbors** if they are next-door neighbors or across-the-street neighbors.

(d) The graph whose vertices are the border terms and whose edges are given by the neighbor relation is called the **border web** of $\mathcal{O}$.

Example 3.2 The border of $\mathcal{O} = \{1, x, y, xy\}$ is $\partial\mathcal{O} = \{x^2, x^2y, xy^2, y^2\}$. Here the border web looks as follows:

(x^2, x^2y) and (y^2, xy^2) are next-door neighbor pairs

(x^2y, xy^2) is an across-the-street neighbor pair.



Proposition 3.3 *The border web is connected.*

Neighbor Syzygies

Definition 3.4 **(a)** For $t, t' \in \mathbb{T}^n$, we call the pair $(\mathrm{lcm}(t,t')/t, -\mathrm{lcm}(t,t')/t')$ the **fundamental syzygy** of (t,t').

(b) The fundamental syzygies of neighboring border terms are also called the **neighbor syzygies**.

Proposition 3.5 **(a)** *Given a tuple of terms* $(t_1, \ldots, t_r)$*, the fundamental syzygies* $\sigma_{ij} = (\mathrm{lcm}(t_i,t_j)/t_i)\, e_i - (\mathrm{lcm}(t_i,t_j)/t_j)\, e_j$ *such that* $1 \le i < j \le r$ *generate the* **syzygy module**

$$\mathrm{Syz}_P(t_1, \ldots, t_r) = \{(f_1, \ldots, f_r) \in P^r \mid f_1 t_1 + \cdots + f_r t_r = 0\}.$$

(b) *The neighbor syzygies generate the module of* **border syzygies** $\mathrm{Syz}_P(b_1, \ldots, b_\nu)$.

Example 3.6 Let us compute the border syzygies for the order ideal $\mathcal{O} = \{1, x, y, xy\}$. We have $\partial\mathcal{O} = \{b_1, b_2, b_3, b_4\}$ with

$$b_1 = x^2,\ b_2 = x^2 y,\ b_3 = xy^2,\ b_4 = y^2$$

and the neighbor pairs (b_1, b_2), (b_2, b_3), (b_3, b_4).

Therefore the border syzygy module $\mathrm{Syz}_P(b_1, b_2, b_3, b_4)$ is generated by the following three neighbor syzygies:

$$\begin{aligned} e_2 - y\, e_1 &= (-y,\ 1,\ 0,\ 0) \\ y\, e_2 - x\, e_3 &= (0,\ y,\ -x,\ 0) \\ e_4 - x\, e_3 &= (0,\ 0,\ -x,\ 1) \end{aligned}$$

4 – Syzygies of Border Bases

Given a choice between two theories,
take the one which is funnier.
(Anonymous)

Goal: Find border basis analogues of Buchberger's Criterion and Schreyer's Theorem!

Given an $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_\nu\}$ as above, we want to define the notion of **lifting syzygies** for them.

Definition 4.1 Let $g_i, g_j \in G$ be two distinct border prebasis polynomials. Then the polynomial

$$S_{ij} = (\mathrm{lcm}(b_i, b_j)/b_i) \cdot g_i - (\mathrm{lcm}(b_i, b_j)/b_j) \cdot g_j$$

is called the **S-polynomial** of g_i and g_j.

Remark 4.2 Let $g_i, g_j \in G$.

(a) If (b_i, b_j) are next-door neighbors with $b_j = x_k\, b_i$ then S_{ij} is of the form $S_{ij} = g_j - x_k\, g_i$.

(b) If (b_i, b_j) are across-the-street neighbors with $x_k\, b_i = x_\ell\, b_j$ then S_{ij} is of the form $S_{ij} = x_k\, g_i - x_\ell\, g_j$.

In both cases we see that the support of S_{ij} is contained in $\mathcal{O} \cup \partial\mathcal{O}$. Hence there exists constants $a_i \in K$ such that the support of

$$\mathrm{NR}_{\mathcal{O},G}(S_{ij}) = S_{ij} - \sum_{m=1}^{\mu} a_m\, g_m \in I$$

is contained in $\mathcal{O}$. If G is a border basis, this implies $\mathrm{NR}_{\mathcal{O},G}(S_{ij}) = 0$.

We shall say that the syzygy $e_j - x_k\, e_i - \sum_{m=1}^{\mu} a_m e_m$ resp. $x_k\; e_i - x_\ell\, e_j - \sum_{m=1}^{\mu} a_m e_m$ is a **lifting** of the neighbor syzygy $e_j - x_k\, e_i$ resp. $x_k\; e_i - x_\ell\, e_j$.

Border Basis Version of Buchberger's Criterion

Theorem 4.3 *(Stetter)*
An $\mathcal{O}$-border prebasis G is an $\mathcal{O}$-border basis if and only if the neighbor syzygies lift, i.e. if and only if we have

$$\mathrm{NR}_{\mathcal{O},G}(S_{ij}) = 0$$

for all (i, j) such that (b_i, b_j) is a pair of neighbors.

Idea of the proof: The vanishing conditions for the normal remainders of the S-polynomials entail certain equalities which have to be satisfied by the coefficients c_{ij} of the border prebasis polynomials. Using a (rather nasty) case-by-case argument, one checks that these are the same equalities that one gets from the conditions that the formal multiplication matrices have to commute.

Example 4.4 Let us look at these conditions for $\mathcal{O} = \{1,\, x,\, y,\, xy\}$. An $\mathcal{O}$-border prebasis $G = \{g_1, g_2, g_3, g_4\}$ is of the form

$$\begin{aligned} g_1 &= x^2 - c_{11} \cdot 1 - c_{21}\, x - c_{31}\, y - c_{41}\, xy \\ g_2 &= x^2 y - c_{12} \cdot 1 - c_{22}\, x - c_{32}\, y - c_{42}\, xy \\ g_3 &= xy^2 - c_{13} \cdot 1 - c_{23}\, x - c_{33}\, y - c_{43}\, xy \\ g_4 &= y^2 - c_{14} \cdot 1 - c_{24}\, x - c_{34}\, y - c_{44}\, xy \end{aligned}$$

The S-polynomials of its neighbor syzygies are

$$\begin{aligned} S_{21} &= g_2 - y g_1 \\ &= -c_{12} - c_{22} x + (c_{11} - c_{32}) y + (c_{21} - c_{42}) xy + c_{31} y^2 + c_{41} xy^2 \\ S_{23} &= y g_2 - x g_3 \\ &= c_{13} x - c_{22} y + (c_{33} - c_{22}) xy + c_{23} x^2 + c_{43} x^2 y - c_{42} xy^2 - c_{32} y^2 \end{aligned}$$

$$
\begin{aligned}
S_{34} &= g_3 - xg_4 \\
&= -c_{13} + (c_{14} - c_{23})x - c_{33}y + (c_{34} - c_{43})xy + c_{24}x^2 + c_{44}x^2y
\end{aligned}
$$

Their normal remainders with respect to G are

$$
\begin{gathered}
\mathrm{NR}_{\mathcal{O},G}(S_{21}) = (-c_{12} + c_{31}c_{14} + c_{41}c_{13}) + (-c_{22} + c_{31}c_{24} + c_{41}c_{23})x \\
+(c_{11} - c_{32} + c_{31}c_{34} + c_{41}c_{33})y + (c_{21} - c_{42} + c_{31}c_{44} + c_{41}c_{43})xy \\
\mathrm{NR}_{\mathcal{O},G}(S_{23}) = (c_{11}c_{23} + c_{12}c_{43} - c_{42}c_{13} - c_{32}c_{14}) + (c_{21}c_{23} + c_{22}c_{43} \\
-c_{42}c_{23} - c_{32}c_{24} + c_{13})x + (-c_{12} + c_{31}c_{23} + c_{32}c_{43} - c_{42}c_{33} - c_{32}c_{34})y \\
+(c_{33} - c_{22} + c_{41}c_{23} - c_{32}c_{44})xy \\
\mathrm{NR}_{\mathcal{O},G}(S_{34}) = (-c_{13} + c_{11}c_{24} + c_{12}c_{44}) + (c_{14} - c_{23} + c_{21}c_{24} + c_{23}c_{44})x \\
+(-c_{33} + c_{31}c_{24} + c_{32}c_{44})y + (c_{34} - c_{43} + c_{41}c_{24} + c_{42}c_{44})xy
\end{gathered}
$$

Here G is a border basis if and only if these 12 coefficients are zero.

Border Basis Version of Schreyer's Theorem

Theorem 4.5 *(Huibregdse)*
Let G be an $\mathcal{O}$-border basis. For every pair (i,j) such that (b_i, b_j) is a neighbor pair, let $s_{ij} = e_j - x_k\, e_i - \sum_{m=1}^{\mu} a_m e_m$ resp.
$s_{ij} = x_k\; e_i - x_\ell\, e_j - \sum_{m=1}^{\mu} a_m e_m$ be the lifting of the corresponding neighbor syzygy.
Then the set $\{s_{ij} \mid (b_i, b_j)$ neighbors$\}$ generates the syzygy module $\mathrm{Syz}_P(g_1, \ldots, g_\nu)$ of the border basis.

Idea of the proof: One has to take an arbitrary syzygy of $(g_1, \ldots, g_\nu)$ and represent it as a linear combination of the syzygies s_{ij}. Unfortunately, in order to reduce the "largest" terms in the syzygy, one may have to introduce even larger terms. A careful analysis of the different cases is necessary to keep the situation under control and make the reduction procedure finite.

Example 4.6 (The Corners of the Unit Square)

Let us have a look the example $\mathbb{X} = \{(\pm 1, \pm 1)\}$. We have $\mathcal{O} = \{1, x, y, xy\}$ and $\mathcal{I}(\mathbb{X}) = \langle g_1, g_2, g_3, g_4 \rangle$ with

$$g_1 = x^2 - 1, \quad g_2 = x^2 y - 1, \quad g_3 = xy^2 - 1, \quad g_4 = y^2 - 1$$

The neighbor syzygies are $e_2 - ye_1$ and $ye_2 - xe_3$ and $e_3 - xe_4$.

The computation of the normal remainders $S_{ij} \longrightarrow \mathrm{NR}_{\mathcal{O},G}(S_{ij})$ shows that the liftings of the neighbor syzygies are

$$\begin{aligned} s_{21} &= e_2 - ye_1 \\ s_{23} &= ye_2 - xe_3 \\ s_{34} &= e_3 - xe_4 \end{aligned}$$

Hence $\mathrm{Syz}_P(g_1, g_2, g_3, g_4)$ is generated by the three tuples $(-y, 1, 0, 0)$, $(0, y, -x, 0)$ and $(0, 0, 1, -x)$.

The Border Basis Scheme

Idea: Use the Commuting Multiplication Matrices / Buchberger Criterion to parametrize $\mathcal{O}$-border bases. Given an $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_\nu\}$ with

$$g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$$

consider the c_{ij} as indeterminates. Let $I_\mathcal{O}$ be the ideal in $K[c_{ij}]$ generated by the entries of all $\mathcal{A}_i \mathcal{A}_j - \mathcal{A}_j \mathcal{A}_i$.

The affine subscheme $\mathbb{B}_\mathcal{O}$ of $K^{\mu\nu}$ defined by $I_\mathcal{O}$ is called the **$\mathcal{O}$-border basis scheme**. Its points correspond 1-1 to the ideals having an $\mathcal{O}$-border basis.

Properties of the Border Basis Scheme

• The scheme $\mathbb{B}_{\mathcal{O}}$ corresponds to an open subset of the **Hilbert scheme** $\mathrm{Hilb}^s(\mathbb{A}^n)$.

• The scheme $\mathbb{B}_{\mathcal{O}}$ comes equipped with a **universal flat family** whose fibers correspond to the rings $P/\langle G\rangle$ where G is an $\mathcal{O}$-border basis.

• A rational curve $C \subseteq \mathbb{B}_{\mathcal{O}}$ which connects the point corresponding to a given ideal $I = \langle G\rangle$ to the point corresponding to the border term ideal $\langle b_1, \ldots, b_\nu\rangle$ is nothing but a **flat deformation** from P/I to $P/\langle b_1, \ldots, b_\nu\rangle$.

5 – Approximate Border Bases

Question to Radio Eriwan:

Is it true that there are polynomials in the approximate vanishing ideal which do not vanish approximately?

Radio Eriwan answers:

In principle yes. Approximately.

Motivation: Suppose we are given some points $\mathbb{X} = \{p_1, \ldots, p_s\}$ in $\mathbb{R}^n$. When does a polynomial **vanish approximately** at $\mathbb{X}$?

Let $\varepsilon > 0$ be a given **threshold number**. We say that $f \in P = \mathbb{R}[x_1, \ldots, x_n]$ **vanishes ε-approximately** at $\mathbb{X}$ if $|f(p_i)| < \varepsilon$ for $i = 1, \ldots, s$.

Problem 1: The polynomials which vanish ε-approximately at $\mathbb{X}$ do not form an ideal!

Problem 2: All polynomials with very small coefficients vanish ε-approximately at $\mathbb{X}$!

Therefore we need to measure the **size** of a polynomial. In other words, we need a topology on $\mathbb{R}[x_1, \ldots, x_n]$.

Definition 5.1 Let $f = a_1 t_1 + \cdots + a_s t_s \in P$, where $a_1, \ldots, a_s \in \mathbb{R} \setminus \{0\}$ and $t_1, \ldots, t_s \in \mathbb{T}^n$. Then the number $\|f\| = \|(a_1, \ldots, a_s)\|$ is called the **(Euclidean) norm** of f.

Clearly, this definition turns P into a normed vector space. Now it is reasonable to consider the condition that polynomials $f \in P$ with $\|f\| = 1$ vanish ε-approximately at $\mathbb{X}$.

Based on these preliminary considerations, we define approximate border bases as follows.

Definition 5.2 Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal, let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$, and let $\varepsilon > 0$.

A set of polynomials $G = \{g_1, \ldots, g_\nu\}$ is called an **ε-approximate $\mathcal{O}$-border basis** if the following conditions are satisfied:

1. For $j = 1, \ldots, \nu$, we have $\|g_j\| = 1$.
2. If a_j denotes the coefficient of b_j in g_j then $|a_j| > \varepsilon$ and $\{\frac{1}{a_j} g_1, \ldots, \frac{1}{a_\nu} g_\nu\}$ is an $\mathcal{O}$-border prebasis.
3. For all pairs (i, j) such that (b_i, b_j) are neighbors, we have $\| \mathrm{NR}_{\mathcal{O},G}(S_{ij})\| < \varepsilon$.

Remark 5.3 If $G = \{g_1, \ldots, g_\nu\}$ is an ε-approximate border basis then the point $(c_{11}, \ldots, c_{\mu\nu})$ in $\mathbb{R}^{\mu\nu}$ given by its coefficients is **close to** the **border basis scheme**.

Example 5.4 Let $\mathcal{O} = \{1,\, x,\, y,\, xy\}$. Then the set

$$g_1 = x^2 + 0.02xy - 0.01y - 1.01 \qquad g_2 = x^2y + 0.03x - 0.98\,y$$

$$g_3 = xy^2 - 1.02\,x \qquad g_4 = y^2 - 0.99$$

is an approximate $\mathcal{O}$-border basis. The ideal $I = \langle g_1, g_2, g_3, g_4\rangle$ is the unit ideal, since $g_3 - x\,g_4 = 0.03x$ shows $-g_1 \equiv 0.01y + 1.01$ and $g_4 \equiv 101^2 - 0.99 (\mathrm{mod}\ I)$.

6 – The AVI Algorithm

A good algorithm is
10% inspiration, 15% perspiration,
and 75% desperation.
(Anonymous)

Goal: Given a set of (empirical) points $\mathbb{X} = \{p_1, \ldots, p_s\}$ in $\mathbb{R}^n$ and $\varepsilon > 0$, find an order ideal $\mathcal{O}$ and an approximate $\mathcal{O}$-border basis G such that the polynomials in G vanish ε-approximately at the points of $\mathbb{X}$.

Notice that, in general,

- we have $\#\mathcal{O} << \#\mathbb{X}$,
- the ideal $\langle G \rangle$ is the unit ideal.

Theorem 6.1 (The Singular Value Decomposition)
Let $\mathcal{A} \in \mathrm{Mat}_{m,n}(\mathbb{R})$*.*

1. *There are orthogonal matrices* $\mathcal{U} \in \mathrm{Mat}_{m,m}(\mathbb{R})$ *and* $\mathcal{V} \in \mathrm{Mat}_{n,n}(\mathbb{R})$ *and a matrix* $\mathcal{S} \in \mathrm{Mat}_{m,n}(\mathbb{R})$ *of the form* $\mathcal{S} = \begin{pmatrix} \mathcal{D} & 0 \\ 0 & 0 \end{pmatrix}$ *such that*

$$\mathcal{A} = \mathcal{U} \cdot \mathcal{S} \cdot \mathcal{V}^{\mathrm{tr}} = \mathcal{U} \cdot \begin{pmatrix} \mathcal{D} & 0 \\ 0 & 0 \end{pmatrix} \cdot \mathcal{V}^{\mathrm{tr}}$$

   where $\mathcal{D} = \mathrm{diag}(s_1, \ldots, s_r)$ *is a diagonal matrix.*

2. *In this decomposition, it is possible to achieve* $s_1 \geq s_2 \geq \cdots \geq s_r > 0$*. The numbers* $s_1, \ldots, s_r$ *depend only on* $\mathcal{A}$ *and are called the* **singular values** *of* $\mathcal{A}$*.*

3. The number r is the rank of $\mathcal{A}$.

4. The matrices $\mathcal{U}$ and $\mathcal{V}$ have the following interpretation:

$$\begin{aligned}
\text{first } r \text{ columns of } \mathcal{U} &\equiv \text{ONB of the column space of } \mathcal{A} \\
\text{last } m-r \text{ columns of } \mathcal{U} &\equiv \text{ONB of the kernel of } \mathcal{A}^{\mathrm{tr}} \\
\text{first } r \text{ columns of } \mathcal{V} &\equiv \text{ONB of the row space of } \mathcal{A} \\
&\equiv \text{ONB of the column space of } \mathcal{A}^{\mathrm{tr}} \\
\text{last } n-r \text{ columns of } \mathcal{V} &\equiv \text{ONB of the kernel of } \mathcal{A}
\end{aligned}$$

Corollary 6.2 *Let* $\mathcal{A} \in \mathrm{Mat}_{m,n}(\mathbb{R})$*, and let* $\varepsilon > 0$ *be given. Let* $k \in \{1, \dots, r\}$ *be chosen such that* $s_k > \varepsilon \geq s_{k+1}$*. Form the matrix* $\widetilde{\mathcal{A}} = \mathcal{U}\,\widetilde{\mathcal{S}}\,\mathcal{V}^{\mathrm{tr}}$ *by setting* $s_{k+1} = \cdots = s_r = 0$ *in* $\mathcal{S}$*. The matrix* $\widetilde{\mathcal{A}}$ *is called the* **singular value truncation** *of* $\mathcal{A}$*.*

1. *We have* $\min\{\|\mathcal{A} - \mathcal{B}\| : \mathrm{rank}(\mathcal{B}) \leq k\} = \|\mathcal{A} - \widetilde{\mathcal{A}}\| = s_{k+1}$*. (Here* $\|\cdots\|$ *denotes the 2-operator norm of a matrix.)*

2. *The vector subspace* $\mathrm{apker}(\mathcal{A}, \varepsilon) = \ker(\widetilde{\mathcal{A}})$ *is the largest dimensional kernel of a matrix whose Euclidean distance from* $\mathcal{A}$ *is at most* ε*. It is called the* **ε-approximate kernel** *of* $\mathcal{A}$*.*

3. *The last* $n - k$ *columns* $v_{k+1}, \dots, v_n$ *of* $\mathcal{V}$ *are an ONB of* $\mathrm{apker}(\mathcal{A}, \varepsilon)$*. They satisfy* $\|\mathcal{A}v_i\| < \varepsilon$*.*

The Stabilized RREF

Given a basis $B = \{f_1, \ldots, f_r\}$ of a vector space V of polynomials and a term ordering σ, we want to find $\mathrm{LT}_{\sigma,\mathrm{ap}}(V)$. We accept only leading terms of unitary polynomials whose leading coefficient is larger than a given threshold $\tau > 0$.

The first step is to transform this to a matrix problem:

Let $S = \mathrm{Supp}(f_1) \cup \cdots \cup \mathrm{Supp}(f_r)$ and write $S = \{t_1, \ldots, t_s\}$ where the terms $t_i \in \mathbb{T}^n$ are ordered such that $t_1 \geq_\sigma t_2 \geq_\sigma \cdots \geq_\sigma t_s$.

Clearly, we have $\mathrm{Supp}(V) \subseteq S$. For $i = 1, \ldots, r$, we write

$$f_i = c_{i1}t_1 + \cdots + c_{is}t_s \qquad \text{with} \quad c_{ij} \in \mathbb{R}$$

Then the matrix $M_{\sigma,B} = (c_{ij}) \in \mathrm{Mat}_{r,s}(\mathbb{R})$ is called the **coefficient matrix** of V with respect to σ and B.

Proposition 6.3 (Stabilized Reduced Row Echelon Form)
Let $A \in \mathrm{Mat}_{m,n}(\mathbb{R})$ and $\tau > 0$ be given. Let $a_1, \ldots, a_n$ be the columns of A. Consider the following instructions.

(1) *Let $\lambda_1 = \|a_1\|$. If $\lambda_1 < \tau$, let $R = (0, \ldots, 0) \in \mathrm{Mat}_{m,1}(\mathbb{R})$. Otherwise, let $Q = ((1/\lambda_1)\, a_1) \in \mathrm{Mat}_{m,1}(\mathbb{R})$ and $R = (\lambda_1, 0, \ldots, 0) \in \mathrm{Mat}_{m,1}(\mathbb{R})$.*

(2) *For $i = 2, \ldots, n$, compute $q_i = a_i - \sum_{j=1}^{i-1} \langle a_i, q_j \rangle\, q_j$ and $\lambda_i = \|q_i\|$. If $\lambda_i < \tau$, append a zero column to R. Otherwise, append the column $(1/\lambda_i)\, q_i$ to Q and the column $(\lambda_i \langle a_1, q_1 \rangle, \ldots, \lambda_i \langle a_{i-1}, q_{i-1} \rangle, \lambda_i, 0, \ldots, 0)$ to R.*

(3) *Starting with the last row and working upwards, use the first non-zero entry of each row of R to clean out the non-zero entries above it.*

(4) *For* $i = 1, \ldots, m$, *compute the norm* ϱ_i *of the* i*-th row of* R. *If* $\varrho_i < \tau$, *set this row to zero. Otherwise, divide this row by* ϱ_i. *Then return the matrix* R.

This is an algorithm which computes a matrix R *in reduced row echelon form. The row space of* R *is contained in the row space of the matrix* $\overline{A}$ *which is obtained from* A *by setting columns whose norm is less than* τ *to zero. Here the pivot elements of* R *are not 1, but its rows are unitary vectors.*

Furthermore, if the rows of A *are unitary and mutually orthogonal, the row vectors of* R *differ by less than* $\tau\, m \sqrt{n}$ *from unitary vectors in the row space of* A.

Theorem 6.4 **(The AVI Algorithm)**
The following algorithm computes an approximate border basis of an approximate vanishing ideal of a finite set of points $\mathbb{X} \subseteq \mathbf{[-1,1]^n}$.

A1 *Start with lists* $G = \emptyset$, $\mathcal{O} = [1]$, *a matrix* $\mathcal{M} = (1,\dots,1)^{\mathrm{tr}} \in \mathrm{Mat}_{s,1}(\mathbb{R})$, *and* $d = 0$.

A2 *Increase* d *by one and let* L *be the list of all terms of degree* d *in* $\partial\mathcal{O}$, *ordered decreasingly w.r.t.* σ. *If* $L = \emptyset$, *return the pair* $(G, \mathcal{O})$ *and stop. Otherwise, let* $L = (t_1, \dots, t_\ell)$.

A3 *Let* m *be the number of columns of* $\mathcal{M}$. *Form the matrix*

$$\mathcal{A} = (\mathrm{eval}(t_1), \dots, \mathrm{eval}(t_\ell), \mathcal{M}) \in \mathrm{Mat}_{s,\ell+m}(\mathbb{R}).$$

Using its **SVD**, *calculate a matrix* $\mathcal{B}$ *whose column vectors are an ONB of the* **approximate kernel** $\mathrm{apker}(\mathcal{A}, \varepsilon)$.

A4 Compute the **stabilized reduced row echelon form** of $\mathcal{B}^{\mathrm{tr}}$ with respect to the given τ. The result is a matrix $\mathcal{C} = (c_{ij}) \in \mathrm{Mat}_{k,\ell+m}(\mathbb{R})$ such that $c_{ij} = 0$ for $j < \nu(i)$. Here $\nu(i)$ denotes the column index of the pivot element in the i^{th} row of $\mathcal{C}$.

A5 For all $j \in \{1, \ldots, \ell\}$ such that there exists a $i \in \{1, \ldots, k\}$ with $\nu(i) = j$ (i.e. for the column indices of the pivot elements), append the polynomial

$$c_{ij} t_j + \sum_{j'=j+1}^{\ell} c_{ij'} t_{j'} + \sum_{j'=\ell+1}^{\ell+m} c_{ij'} u_{j'}$$

to the list G, where $u_{j'}$ is the $(j' - \ell)^{\mathrm{th}}$ element of $\mathcal{O}$.

A6 For all $j = \ell, \ell - 1, \ldots, 1$ such that the j^{th} column of $\mathcal{C}$ contains no pivot element, append the term t_j as a new first element to $\mathcal{O}$ and append the column $\mathrm{eval}(t_j)$ as a new first column to $\mathcal{M}$.

A7 Using the SVD of $\mathcal{M}$, calculate a matrix $\mathcal{B}$ whose column vectors are an ONB of $\operatorname{apker}(\mathcal{M}, \varepsilon)$.

A8 Repeat steps **A4** – **A7** until $\mathcal{B}$ is empty. Then continue with step **A2**.

The resulting set $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ contains an order ideal of terms such that there is no unitary polynomial in $\langle \mathcal{O} \rangle_K$ which vanishes ε-approximately on $\mathbb{X}$.

The resulting set G is a δ-approximate $\mathcal{O}$-border basis. (An explicit bound for δ can be given.)

The AVI algorithm has been implemented in the **ApCoCoA** library

(see `http://www.apcocoa.org`)

Let us compute some examples with the AVI algorithm.

Example 6.5 **(Four Almost Aligned Points)**

Let $\mathbb{X} = \{(0, 0.01), (0.34, 0.32), (0.65, 0.68), (0.99, 1)\}$, and let $\varepsilon = 0.05$. When we apply the AVI algorithm to this case, we get $\mathcal{O} = \{1, y, y^2\}$ and the approximate border basis $G = \{0.71x - 0.70y, 0.70xy - 0.71y^2 + 0.02y, 0.51xy^2 - 0.80y^2 + 0.31y - 0.01, 0.51y^3 - 0.80y^2 + 0.30y - 0.01\}$.

At first glance, it may be surprising that $\mathcal{O}$ corresponds to only three points. Which points are these? In the next section, we shall show that they are $\mathbb{X}' = \{(0.03, 0.04), (0.52, 0.52), (0.97, 0.98)\}$.

What has happened is that AVI found a curve of degree 3 passing close to $\mathbb{X}$, namely $g_4 = 0.51y^3 - 0.80y^2 + 0.30y - 0.01$.

Now let us look at the Example presented by C. Fassino where the SOI and NBM algorithms find no stable border basis.

Example 6.6 **(Five Points on Two Conics and a Cubic)**

Let $\mathbb{X} = \{(0,1), (0.2, 0.4), (0.28, 0.28), (0.4, 0.2), (1, 0)\}$.

(a) When we choose $\varepsilon = 0.1$, we get $\mathcal{O} = \{1, x, y\}$ and the (unitary) approximate border basis $G = 0.52x^2 - 0.77x - 0.25y + 0.25,\ 0.94xy + 0.18x + 0.18y - 0.18,\ 0.51y^2 - 0.26x - 0.77y + 0.26\}$. The set G approximately defines $\mathbb{X}' = \{(0, 0.98), (0.28, 0.29), (0.98, 0)\}$.

(b) The choice $\varepsilon = 0.01$ leads to $\mathcal{O}' = \{1, x, y, y^2\}$ and $G' = \{0.3x^2+0.3y^2-0.6x-0.6y+0.3,\ 0.94xy+0.18x+0.18y-0.18,\ 0.95xy^2+0.19y^2 - 0.03x - 0.22y + 0.03,\ 0.42y^3 - 0.77y^2 + 0.10x + 0.44y - 0.10\}$. The set G' approximately defines the four points $\mathbb{X}'' = \{(0, 0.99), (0.21, 0.37), (0.37, 0.21), (0.99, 0)\}$. Thus even this small choice of ε leads to a decrease in the codimension of $\mathcal{I}(\mathbb{X})$.

Finally we consider an industrial application of the AVI algorithm.

Example 6.7 Suppose that $\mathbb{X}$ consists of 6000 points in $\mathbb{R}^5$. The relative errors in the coordinates of the points of $\mathbb{X}$ are about 10%. It is reasonable to use the AVI algorithm with $\varepsilon = 0.1$.

The result (after ca. 0.5s CPU time on my laptop) is the order ideal $\mathcal{O} = \{1, x[5], x[4], x[3], x[2], x[1], x[5]^2, x[4]x[5], x[3]x[5], x[2]x[5], x[1]x[5], x[4]^2, x[3]x[4], x[2]x[4], x[1]x[4], x[3]^2, x[2]x[3], x[5]^3, x[4]x[5]^2, x[3]x[5]^2\}$ consisting of only **20** terms. The approximate border basis G has 43 elements.

It is **not true** that there are 20 real points such that G is an approximate border basis of their vanishing ideal.

One can only say that the 20-dimensional space $\langle \mathcal{O} \rangle_{\mathbb{R}}$ suffices to interpolate approximately at the given 6000 points.

7 – Rational Recovery

I'm an excellent housekeeper.
Every time I get a divorce,
I keep the house.
(Zsa Zsa Gabor)

In this last part we want to address the following

Rational Recovery Problem: Given an approximate border basis, find an exact border basis, defined over $\mathbb{Q}$, which is "close-by" in the sense that its coefficients differ very little from the approximate border basis.

Proposition 7.1 **(Properties of Multiplication Matrices)**

Let $\mathbb{X} \subset \mathbb{R}^n$ be a finite point set, and let $\mathcal{O}$ be an order ideal such that there exists an $\mathcal{O}$-border basis of $\mathcal{I}(\mathbb{X})$.

(a) *The eigenvalues of the i-th multiplication matrix M_{x_i} w.r.t. $\mathcal{O}$ are the i-th coordinates of the points of $\mathbb{X}$.*

(b) *Let $\mathcal{O} = \{1, x_1, \ldots, x_n, t_{n+2}, \ldots, t_\mu\}$. Then the joint eigenvectors of $M_{x_1}^{\mathrm{tr}}, \ldots, M_{x_n}^{\mathrm{tr}}$ are of the form $v_i = (1, p_{i1}, \ldots, p_{in}, \ldots)$ where the points $(p_{i1}, \ldots, p_{in})$ are the points of $\mathbb{X}$.*

$\mathcal{O} = \{t_1, \ldots, t_\mu\}$ order ideal

$\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ border of $\mathcal{O}$

$G = \{g_1, \ldots, g_\nu\}$ an ε-approximate $\mathcal{O}$-border basis

The formal multiplication matrices $M_{x_1}, \ldots, M_{x_n}$ associated to G are also called the **approximate multiplication matrices**.

Remark 7.2 The approximate multiplication matrices commute approximately, i.e. the entries of their commutators are smaller than ε.

Question: How can we find "approximate" joint eigenvectors of $M_{x_1}^{\mathrm{tr}}, \ldots, M_{x_n}^{\mathrm{tr}}$?

The Rational Recovery Algorithm

In the above setting we proceed as follows:

(1) Form a generic linear combination $M_\ell = \ell_1 M_{x_1} + \cdots + \ell_n M_{x_n}$ of $M_{x_1}, \ldots, M_{x_n}$.

Then M_ℓ is the matrix of the multiplication by the generic linear form $\ell = \ell_1 x_1 + \cdots + \ell_n x_n$. The eigenvalues of M_ℓ are "as separated as possible".

(2) Compute $\mathbb{Q}$-rational points $p_1, \ldots, p_\mu$ by finding the eigenvectors of M_ℓ^{tr}, writing $v_i = (1, \tilde{p}_{i1}, \ldots, \tilde{p}_{in}, \ldots)$, and considering the floating point numbers $\tilde{p}_{ij}$ as rationals.

(3) Calculate the exact $\mathcal{O}$-border basis of the vanishing ideal of $\mathbb{X}' = \{p_1, \ldots, p_\mu\}$ over $\mathbb{Q}[x_1, \ldots, x_n]$.

Some examples for rational recovery were already mentioned above. Let us go carefully through one more.

Example 7.3 (The Corners of the Unit Square)

Let $\mathbb{X} = \{(0.02, 0.01), (0.99, 0.01), (1.01, 0.98), (0.02, 0.99)\}$ be the slightly perturbed corners of the unit square.

We apply the AVI algorithm with $\varepsilon = 0.05$ and get the order ideal $\mathcal{O} = \{1,\ x,\ y,\ xy\}$ and the approximate (unitary) $\mathcal{O}$-border basis $G = \{g_1, g_2, g_3, g_4\}$ with

$$\begin{aligned} g_1 &= 0.70x^2 - 0.01xy - 0.71x + 0.01 \\ g_2 &= 0.70y^2 - 0.70y \\ g_3 &= 0.69x^2y - 0.71xy + 0.01y \\ g_4 &= 0.71xy^2 - 0.70xy \end{aligned}$$

Therefore the multiplication matrices are

$$M_x = \begin{pmatrix} 0 & -0.01 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.02 \\ 0 & 0.02 & 1 & 1.03 \end{pmatrix} \text{ and } M_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0.98 \end{pmatrix}$$

Both matrices have two eigenvalues very close to 0 and two eigenvalues very close to 1. If we try to compute the joint eigenvectors of M_x^{tr} and M_y^{tr}, we get numerically unstable results.

Therefore we form a "random" linear combination $M_\ell = 0.3M_x + 0.5M_x$ and compute the eigenvectors of M_ℓ^{tr}. The result is $v_1 = (1, 0.01, 0, 0)$, $v_2 = (1, 0.99, 0, 0)$, $v_3 = (1, 0.02, 0.99, 0.01)$, $v_4 = (1, 0.99, 0.98, 0.97)$.

This yields the exact points
$\mathbb{X}' = \{(0.01, 0), (0.99, 0), (0.02, 0.99), (0.99, 0.98)\}$ which are in good agreement with the input points.

Finally, the exact $\mathcal{O}$-border basis of $\mathbb{X}'$ is $G' = \{g_1', g_2', g_3', g_4'\}$ where

$$\begin{aligned}
g_1' &= x^2 - 1/99xy - x + 1/100y + 99/10000 \\
g_2' &= y^2 + 1/97xy - 1921/1940y \\
g_3' &= x^2y - 101/100xy + 99/5000y \\
g_4' &= xy^2 - 2376/2425xy - 99/485000y
\end{aligned}$$

It is easy to check that this is again in good agreement with G.

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In **THE END**,

everything is a gag.

(Charlie Chaplin)

Thank you for your attention!