Exact and Approximate Border Bases

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March 4, 2008

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1 – Border Bases

I don't make jokes.
I just watch the government and report the facts.
(Will Rogers)

K field

$P = K[x_1, \ldots, x_n]$ polynomial ring over K

$I \subseteq P$ zero-dimensional polynomial ideal (i.e. $\dim_K(P/I) < \infty$)

$\mathbb{T}^n = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_i \geq 0\}$ monoid of terms

Motivation

Goal: We are looking for a set of terms $\mathcal{O}$ whose residue classes form a K-vector space basis of P/I.

Example 1.1 Given a term ordering σ, the set $\mathbb{T}^n \setminus \mathrm{LT}_\sigma(I)$ is a K-vector space basis of P/I by Macaulay's Basis Theorem.

Question: Are there other suitable sets $\mathcal{O}$?

Idea: The algebra structure of P/I can be described by specifying the **multiplication matrices**, i.e. the matrices A_i of the multiplication maps $\mu_{x_i} : P/I \longrightarrow P/I$ with respect to the basis $\mathcal{O}$.

Therefore we need to fix how a term b_j in the

border $\quad \partial\mathcal{O} = (x_1\mathcal{O} \cup \cdots \cup x_n\mathcal{O}) \setminus \mathcal{O}$

of $\mathcal{O}$ is rewritten as a linear combination of the terms in $\mathcal{O}$.

Thus, for every $b_j \in \partial\mathcal{O}$, a polynomial of the form

$$g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$$

with $c_{ij} \in K$ and $t_i \in \mathcal{O}$ should be contained in I.

Moreover, we would not like that $x_k g_j \in I$. Hence we want $x_k b_j \notin \mathcal{O}$. Therefore the set $\mathbb{T}^n \setminus \mathcal{O}$ should be a monoideal.

Consequently, $\mathcal{O}$ should be an **order ideal**, that is it should be closed under forming divisors: $t \in \mathcal{O}$ and $t' \mid t$ implies $t' \in \mathcal{O}$.

Definition of Border Bases

Definition 1.2 **(a)** A (finite) set $\mathcal{O} \subset \mathbb{T}^n$ is called an **order ideal** if $t \in \mathcal{O}$ and $t' \mid t$ implies $t' \in \mathcal{O}$.

(b) Let $\mathcal{O}$ be an order ideal. The set $\partial\mathcal{O} = (x_1\mathcal{O} \cup \cdots \cup x_n\mathcal{O}) \setminus \mathcal{O}$ is called the **border** of $\mathcal{O}$.

(c) Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal and $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ its border. A set of polynomials $\{g_1, \ldots, g_\nu\} \subset I$ of the form

$$g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$$

with $c_{ij} \in K$ and $t_i \in \mathcal{O}$ is called an **$\mathcal{O}$-border prebasis** of I.

(d) An $\mathcal{O}$-border prebasis of I is called an **$\mathcal{O}$-border basis** of I if the residue classes of the terms in $\mathcal{O}$ are a K-vector space basis of P/I.

2 – The Running Example

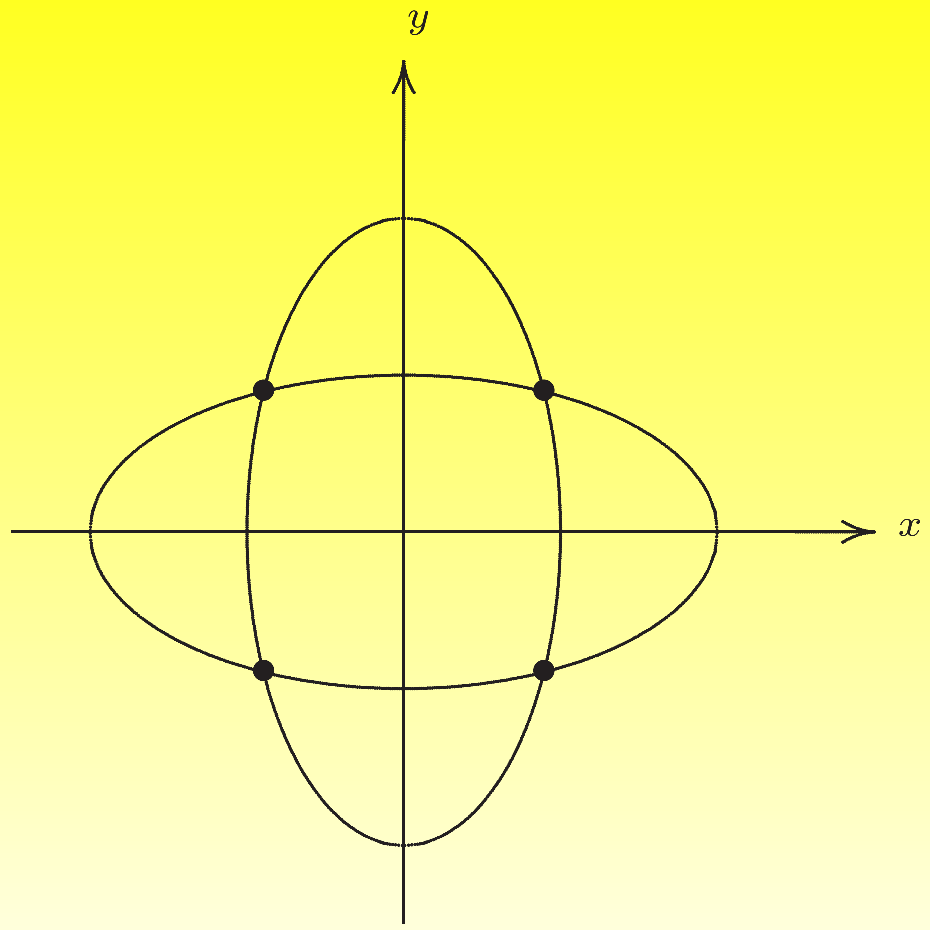
It is difficult to get a man to understand something when his salary depends upon his not understanding it.

(Upton Sinclair)

Example 2.1 In the ring $P = \mathbb{R}[x,y]$, consider the ideal $I = \langle f_1, f_2 \rangle$ where

$$\begin{aligned} f_1 &= \tfrac{1}{4}x^2 + y^2 - 1 \\ f_2 &= x^2 + \tfrac{1}{4}y^2 - 1 \end{aligned}$$

The zero set of I in $\mathbb{A}^2(\mathbb{R})$ consists of the four points $\mathbb{X} = \{(\pm\sqrt{0.8}, \pm\sqrt{0.8})\}$.


y
x

We use $\sigma =$ `DegRevLex` and compute $\mathrm{LT}_\sigma(I) = \langle x^2, y^2 \rangle$. Therefore the order ideal

$$\mathcal{O} = \{1, x, y, xy\}$$

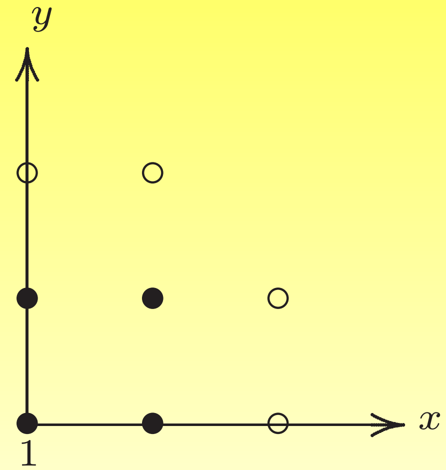
represents a basis of P/I. Its border is

$$\partial\mathcal{O} = \{x^2, x^2y, xy^2, y^2\}.$$

A border basis of I is given by $G = \{g_1, g_2, g_3, g_4\}$ where

$$\begin{aligned} g_1 &= x^2 - 0.8 \\ g_2 &= x^2y - 0.8\,y \\ g_3 &= xy^2 - 0.8\,x \\ g_4 &= y^2 - 0.8 \end{aligned}$$

A Picture of this Order Ideal and its Border



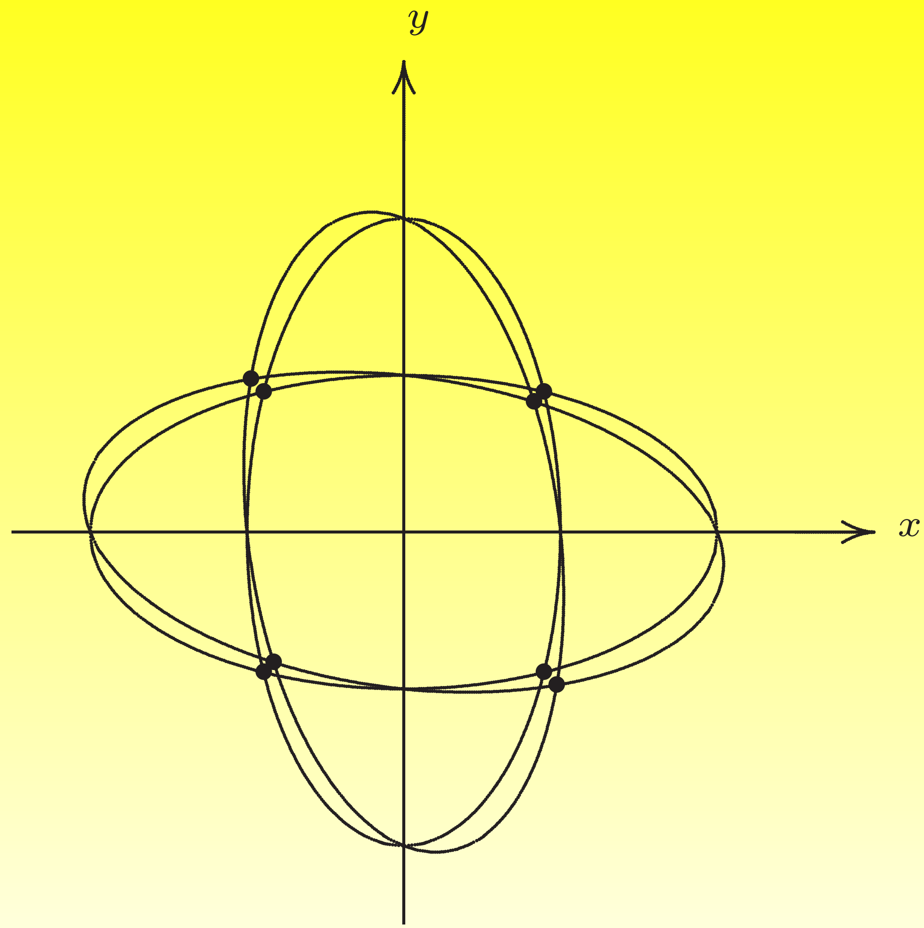
3 – The Disturbed Example

Change is inevitable,
except from a vending machine.
(Anonymous)

Example 3.1 Now we consider the slightly changed ideal $\tilde{I} = \langle \tilde{f}_1, \tilde{f}_2 \rangle$ where

$$\begin{aligned} \tilde{f}_1 &= 0.25\,x^2 + y^2 + 0.01\,xy - 1 \\ \tilde{f}_2 &= x^2 + 0.25\,y^2 + 0.01\,xy - 10 \end{aligned}$$

Its zero set consists of four perturbed points $\widetilde{\mathbb{X}}$ close to those in $\mathbb{X}$.


y
x

The ideal $\tilde{I} = \langle \tilde{f}_1,\, \tilde{f}_2 \rangle$ has the reduced σ-Gröbner basis

$$\{x^2 - y^2,\, xy + 125\, y^2 - 100,\, y^3 - \tfrac{25}{3906}\, x + \tfrac{3125}{3906}\, y\}$$

Moreover, we have $\mathrm{LT}_\sigma(\tilde{I}) = \langle x^2,\, xy,\, y^3 \rangle$ and
$\mathbb{T}^2 \setminus \mathrm{LT}_\sigma\{\tilde{I}\} = \{1,\, x,\, y,\, y^2\}$.

A **small** change in the coefficients of f_1 and f_2 has led to a **big** change in the Gröbner basis of $\langle f_1, f_2 \rangle$ and in the associated vector space basis of $\mathbb{R}[x, y]/\langle f_1, f_2 \rangle$, although the zeros of the ideal have not changed much. Numerical analysts call this kind of unstable behavior a **representation singularity**.

However, also the ideal $\tilde{I}$ has a a border basis with respect to $\mathcal{O} = \{1, x, y, xy\}$. Recall that the border of $\mathcal{O}$ is $\partial\mathcal{O} = \{x^2, x^2y, xy^2, y^2\}$.

The $\mathcal{O}$-border basis of $\tilde{I}$ is $\tilde{G} = \{\tilde{g}_1,\, \tilde{g}_2,\, \tilde{g}_3,\, \tilde{g}_4\}$ where

$$\begin{aligned}
\tilde{g}_1 &= x^2 + 0.008\,xy - 0.8 \\
\tilde{g}_2 &= x^2y + \tfrac{25}{3906}\,x - \tfrac{3125}{3906}\,y \\
\tilde{g}_3 &= xy^2 - \tfrac{3125}{3906}\,x + \tfrac{25}{3906}\,y \\
\tilde{g}_4 &= y^2 + 0.008\,xy - 0.8\}
\end{aligned}$$

When we vary the coefficients of xy in the two generators from zero to 0.01, we see that one border bases changes continuously into the other. Thus the border basis behaves numerically stable under small perturbations of the coefficient of xy.

4 – Properties of Border Bases

The list of the theorems I knew
made limericks end at line two.
(Anonymous)

In the following, we use the following notation:

$\mathcal{O} = \{t_1, \ldots, t_\mu\}$ order ideal

$\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ border of $\mathcal{O}$

$G = \{g_1, \ldots, g_\nu\}$ is an $\mathcal{O}$-border prebasis, where

$g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$ with $c_{ij} \in K$

$I = \langle g_1, \ldots, g_\nu \rangle$ ideal generated by G

Proposition 4.1 (Existence of Border Bases)

(a) *The ideal I need not have an $\mathcal{O}$-border basis. But if it does, the $\mathcal{O}$-border basis of I is uniquely determined.*

(b) *If $\mathcal{O}$ is of the form $\mathbb{T}^n \setminus \mathrm{LT}_\sigma(I)$ for some term ordering σ, then I has an $\mathcal{O}$-border basis. It contains the reduced σ-Gröbner basis of I.*

Proposition 4.2 *There exists a* **Division Algorithm** *for border prebases.*

Proposition 4.3 *The rewriting system defined by the rules $b_j \longrightarrow \sum_{i=1}^{\mu} c_{ij} t_i$ is confluent. (But it is in general not* **terminating***, i.e. not Noetherian.)*

Characterization Using Multiplication Matrices

For $r \in \{1, \ldots, n\}$, we define the r-th **formal multiplication matrix** $\mathcal{A}_r$ as follows:

Multiply $t_i \in \mathcal{O}$ by x_r. If $x_r t_i = b_j$ is in the border of $\mathcal{O}$, rewrite it using the prebasis polynomial $g_j = b_j - \sum_{k=1}^{\mu} c_{kj} t_k$ and put $(c_1, \ldots, c_\mu)$ into the i-th column of $\mathcal{A}_r$. But if $x_r t_i = t_j$ then put the j-th unit vector into the i-th column of $\mathcal{A}_r$.

Theorem 4.4 *(Mourrain)*
The set G is the $\mathcal{O}$-border basis of I if and only if the formal multiplication matrices commute, i.e. iff

$$\mathcal{A}_i \, \mathcal{A}_j = \mathcal{A}_j \, \mathcal{A}_i \qquad \text{for } 1 \leq i < j \leq n.$$

5 – Neighbors

Under capitalism, man exploits man.
Under communism, it's just the opposite.
(John Kenneth Galbraith)

Definition 5.1 Let $b_i, b_j \in \partial\mathcal{O}$ be two distinct border terms.

(a) The border terms b_i and b_j are called **next-door neighbors** if $b_i = x_k\, b_j$ for some $k \in \{1, \ldots, n\}$.

(b) The border terms b_i and b_j are called **across-the-street neighbors** if $x_k\, b_i = x_\ell\, b_j$ for some $k, \ell \in \{1, \ldots, n\}$.

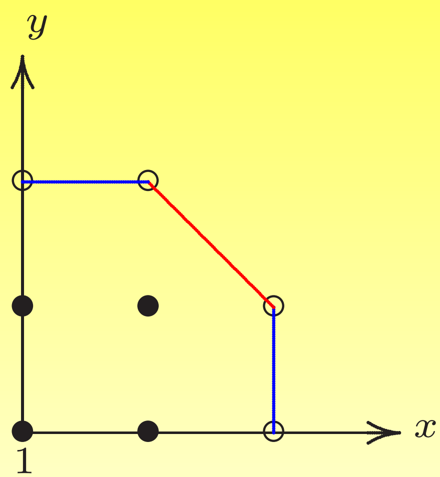
(c) The border terms b_i and b_j are called **neighbors** if they are next-door neighbors or across-the-street neighbors.

(d) The graph whose vertices are the border terms and whose edges are given by the neighbor relation is called the **border web** of $\mathcal{O}$.

Example 5.2 The border of $\mathcal{O} = \{1,\, x,\, y,\, xy\}$ is $\partial\mathcal{O} = \{x^2,\, x^2y,\, xy^2,\, y^2\}$. Here the border web look as follows:

$(x^2,\, x^2y)$ and (y^2, xy^2) are next-door neighbor pairs

(x^2y, xy^2) is an across-the-street neighbor pair



Proposition 5.3 *The border web is connected.*

Neighbor Syzygies

Definition 5.4 **(a)** For $t, t' \in \mathbb{T}^n$, we call the pair $(\mathrm{lcm}(t,t')/t, -\mathrm{lcm}(t,t')/t')$ the **fundamental syzygy** of (t,t').

(b) The fundamental syzygies of neighboring border terms are also called the **neighbor syzygies**.

Proposition 5.5 **(a)** *Given a tuple of terms* $(t_1,\dots,t_r)$, *the fundamental syzygies* $\sigma_{ij} = (\mathrm{lcm}(t_i,t_j)/t_i)\, e_i - (\mathrm{lcm}(t_i,t_j)/t_j)\, e_j$ *such that* $1 \le i < j \le r$ *generate the* **syzygy module**

$$\mathrm{Syz}_P(t_1,\dots,t_r) = \{(f_1,\dots,f_r) \in P^r \mid f_1t_1 + \cdots + f_rt_r = 0\}.$$

(b) *The neighbor syzygies generate the module of* **border syzygies** $\mathrm{Syz}_P(b_1,\dots,b_\nu)$.

Example 5.6 Let us compute the border syzygies for the order ideal $\mathcal{O} = \{1,\, x,\, y,\, xy\}$. We have $\partial\mathcal{O} = \{b_1, b_2, b_3, b_4\}$ with

$$b_1 = x^2,\; b_2 = x^2 y,\; b_3 = xy^2,\; b_4 = y^2$$

and the neighbor pairs (b_1, b_2), (b_2, b_3), (b_3, b_4).

Therefore the border syzygy module $\mathrm{Syz}_P(b_1, b_2, b_3, b_4)$ is generated by the following three neighbor syzygies:

$$\begin{aligned} e_2 - y\, e_1 &= (-y,\, 1,\, 0,\, 0) \\ y\, e_2 - x\, e_3 &= (0,\, y,\, -x,\, 0) \\ e_4 - x\, e_3 &= (0,\, 0,\, -x,\, 1) \end{aligned}$$

6 – Syzygies of Border Bases

Given a choice between two theories,
take the one which is funnier.
(Anonymous)

Goal: Find border basis analogues of Buchberger's Criterion and Schreyer's Theorem!

Given an $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_\nu\}$ as above, we want to define the notion of **lifting syzygies** for them.

Definition 6.1 Let $g_i, g_j \in G$ be two distinct border prebasis polynomials. Then the polynomial

$$S_{ij} = (\mathrm{lcm}(b_i, b_j)/b_i) \cdot g_i - (\mathrm{lcm}(b_i, b_j)/b_j) \cdot g_j$$

is called the **S-polynomial** of g_i and g_j.

Remark 6.2 Let $g_i, g_j \in G$.

(a) If (b_i, b_j) are next-door neighbors with $b_j = x_k\, b_i$ then S_{ij} is of the form $S_{ij} = g_j - x_k\, g_i$.

(b) If (b_i, b_j) are across-the-street neighbors with $x_k\, b_i = x_\ell\, b_j$ then S_{ij} is of the form $S_{ij} = x_k\, g_i - x_\ell\, b_j$.

In both cases we see that the support of S_{ij} is contained in $\mathcal{O} \cup \partial\mathcal{O}$. Hence there exists constants $a_i \in K$ such that the support of

$$\mathrm{NR}_{\mathcal{O},G}(S_{ij}) = S_{ij} - \sum_{m=1}^{\mu} a_m\, g_m \in I$$

is contained in $\mathcal{O}$. If G is a border basis, this implies $\mathrm{NR}_{\mathcal{O},G}(S_{ij}) = 0$.

We shall say that the syzygy $e_j - x_k\, e_i - \sum_{m=1}^{\mu} a_m e_m$ resp. $x_k\ e_i - x_\ell\, e_j - \sum_{m=1}^{\mu} a_m e_m$ is a **lifting** of the neighbor syzygy $e_j - x_k\, e_i$ resp. $x_k\ e_i - x_\ell\, e_j$.

Border Basis Version of Buchberger's Criterion

Theorem 6.3 *(Stetter)*
An $\mathcal{O}$-border prebasis G is an $\mathcal{O}$-border basis if and only if the neighbor syzygies lift, i.e. if and only if we have

$$\mathrm{NR}_{\mathcal{O},G}(S_{ij}) = 0$$

for all (i, j) such that (b_i, b_j) is a pair of neighbors.

Idea of the proof: The vanishing conditions for the normal remainders of the S-polynomials entail certain equalities which have to be satisfied by the coefficients c_{ij} of the border prebasis polynomials. Using a (rather nasty) case-by-case argument, one checks that these are the same equalities that one gets from the conditions that the formal multiplication matrices have to commute.

Example 6.4 Let us look at these conditions for $\mathcal{O} = \{1,\, x,\, y,\, xy\}$. An $\mathcal{O}$-border prebasis $G = \{g_1, g_2, g_3, g_4\}$ is of the form

$$
\begin{aligned}
g_1 &= x^2 - c_{11}\cdot 1 - c_{12}\,x - c_{13}\,y - c_{14}\,xy \\
g_2 &= x^2y - c_{21}\cdot 1 - c_{22}\,x - c_{23}\,y - c_{24}\,xy \\
g_3 &= xy^2 - c_{31}\cdot 1 - c_{32}\,x - c_{33}\,y - c_{34}\,xy \\
g_4 &= y^2 - c_{41}\cdot 1 - c_{42}\,x - c_{43}\,y - c_{44}\,xy
\end{aligned}
$$

The S-polynomials of its neighbor syzygies are

$$
\begin{aligned}
S_{21} &= g_2 - yg_1 \\
&= -c_{21} - c_{22}x + (c_{11} - c_{23})y + (c_{12} - c_{24})xy + c_{13}y^2 + c_{14}xy^2 \\
S_{23} &= yg_2 - xg_3 \\
&= c_{31}x - c_{22}y + (c_{33} - c_{22})xy + c_{32}x^2 + c_{34}x^2y - c_{24}xy^2 - c_{23}y^2
\end{aligned}
$$

$$\begin{aligned} S_{34} &= g_3 - xg_4 \\ &= -c_{31} + (c_{41} - c_{32})x - c_{33}y + (c_{43} - c_{34})xy + c_{42}x^2 + c_{44}x^2y \end{aligned}$$

Their normal remainders with respect to G are

$$\mathrm{NR}_{\mathcal{O},G}(S_{21}) = (-c_{21} + c_{13}c_{41} + c_{14}c_{31}) + (-c_{22} + c_{13}c_{42} + c_{14}c_{32})x$$
$$+(c_{11} - c_{23} + c_{13}c_{43} + c_{14}c_{33})y + (c_{12} - c_{24} + c_{13}c_{44} + c_{14}c_{34})xy$$
$$\mathrm{NR}_{\mathcal{O},G}(S_{23}) = (c_{11}c_{32} + c_{21}c_{34} - c_{24}c_{31} - c_{23}c_{41}) + (c_{12}c_{32} + c_{22}c_{34}$$
$$-c_{24}c_{32} - c_{23}c_{42} + c_{31})x + (-c_{21} + c_{13}c_{32} + c_{23}c_{34} - c_{24}c_{33} - c_{23}c_{43})y$$
$$+(c_{33} - c_{22} + c_{14}c_{32} - c_{23}c_{44})xy$$
$$\mathrm{NR}_{\mathcal{O},G}(S_{34}) = (-c_{31} + c_{11}c_{42} + c_{21}c_{44}) + (c_{41} - c_{32} + c_{12}c_{42} + c_{32}c_{44})x$$
$$+(-c_{33} + c_{13}c_{42} + c_{23}c_{44})y + (c_{43} - c_{34} + c_{14}c_{42} + c_{24}c_{44})xy$$

Here G is a border basis if and only if these 12 coefficients are zero.

Border Basis Version of Schreyer's Theorem

Theorem 6.5 *(Huibregdse)*
Let G be an $\mathcal{O}$-border basis. For every pair (i, j) such that (b_i, b_j) is a neighbor pair, let $s_{ij} = e_j - x_k\, e_i - \sum_{m=1}^{\mu} a_m e_m$ resp.
$s_{ij} = x_k\; e_i - x_\ell\, e_j - \sum_{m=1}^{\mu} a_m e_m$ be the lifting of the corresponding neighbor syzygy.
Then the set $\{s_{ij} \mid (b_i, b_j) \text{ neighbors}\}$ generates the syzygy module $\mathrm{Syz}_P(g_1, \ldots, g_\nu)$ of the border basis.

Idea of the proof: One has to take an arbitrary syzygy of $(g_1, \ldots, g_\nu)$ and represent it as a linear combination of the syzygies s_{ij}. Unfortunately, in order to reduce the "largest" terms in the syzygy, one may have to introduce even larger terms. A careful analysis of the different cases is necessary to keep the situation under control and make the reduction procedure finite.

Example 6.6 Let us have yet another look at the example where $\mathcal{O} = \{1, x, y, xy\}$ and $I = \langle g_1, g_2, g_3, g_4 \rangle$ with

$$g_1 = x^2 - 0.8 \qquad g_2 = x^2 y - 0.8\, y$$

$$g_3 = xy^2 - 0.8\, x \qquad g_4 = y^2 - 0.8$$

The neighbor syzygies are $e_2 - ye_1$ and $ye_2 - xe_3$ and $e_3 - xe_4$.

The computation of the normal remainders $S_{ij} \longrightarrow \mathrm{NR}_{\mathcal{O},G}(S_{ij})$ shows that the liftings of the neighbor syzygies are

$$\begin{aligned} s_{21} &= e_2 - ye_1 \\ s_{23} &= ye_2 - xe_3 \\ s_{34} &= e_3 - xe_4 \end{aligned}$$

Hence $\mathrm{Syz}_P(g_1, g_2, g_3, g_4)$ is generated by the three tuples $(-y, 1, 0, 0)$, $(0, y, -x, 0)$ and $(0, 0, 1, -x)$.

7 – Approximate Border Bases

The following is the result of exhaustive research,
careful analysis, and prolonged deliberation ...
after which I flipped a coin. (Anonymous)

Motivation: Suppose we are given some points $\mathbb{X} = \{p_1, \ldots, p_s\}$ in $\mathbb{R}^n$. When does a polynomial **vanish approximately** at $\mathbb{X}$?

Let $\varepsilon > 0$ be a given **threshold number**. We say that $f \in P = \mathbb{R}[x_1, \ldots, x_n]$ **vanishes ε-approximately** at $\mathbb{X}$ if $|f(p_i)| < \varepsilon$ for $i = 1, \ldots, s$.

Problem 1: The polynomials which vanish ε-approximately at $\mathbb{X}$ do not form an ideal!

Problem 2: All polynomials with very small coefficients vanish ε-approximately at $\mathbb{X}$!

Therefore we need to measure the **size** of a polynomial. In other words, we need a topology on $\mathbb{R}[x_1, \ldots, x_n]$.

Definition 7.1 Let $f = a_1 t_1 + \cdots + a_s t_s \in P$, where $a_1, \ldots, a_s \in \mathbb{R} \setminus \{0\}$ and $t_1, \ldots, t_s \in \mathbb{T}^n$. Then the number $\|f\| = \|(a_1, \ldots, a_s)\|$ is called the **(Euclidean) norm** of f.

Clearly, this definition turns P into a normed vector space. Now it is reasonable to consider the condition that polynomials $f \in P$ with $\|f\| = 1$ vanish ε-approximately at $\mathbb{X}$.

Definition 7.2 An ideal $I \subseteq P$ is called an **ε-approximate vanishing ideal** of $\mathbb{X}$ if there exists a system of generators $\{f_1, \ldots, f_r\}$ of I such that $\|f_i\| = 1$ and f_i vanishes ε-approximately at $\mathbb{X}$ for $i = 1, \ldots, r$.

Based on these preliminary consideratons, we define approximate border bases as follows.

Definition 7.3 Let $I \subset P$ be a zero-dimensional ideal, let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal containing $\dim_{\mathbb{R}}(P/I)$ elements, let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$, and let $\varepsilon > 0$.

A set of polynomials $G = \{g_1, \ldots, g_\nu\}$ is called an **ε-approximate $\mathcal{O}$-border basis** of I if the following conditions are satisfied:

1. For $j = 1, \ldots, \nu$, we have $\|g_j\| = 1$.
2. If a_j denotes the coefficient of b_j in g_j then $|a_j| > \varepsilon$ and $\{\frac{1}{a_j}\, g_1, \ldots, \frac{1}{a_\nu}\, g_\nu\}$ is an $\mathcal{O}$-border prebasis.
3. For all pairs (i, j) such that (b_i, b_j) are neighbors, we have $\|\operatorname{NR}_{\mathcal{O},G}(S_{ij})\| < \varepsilon$.

Remark 7.4 **(a)** Given a set of points $\mathbb{X}$ in $\mathbb{Q}^n$ and a threshold number $\varepsilon > 0$, there exist algorithms for computing an ε-approximate border basis of an ε-approximate vanishing ideal of $\mathbb{X}$.

(b) Given unitary **empirical** polynomials $f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_n]$ which are **close to** generating a zero-dimensional ideal I, there exists an algorithm which computes an ε-approximate border basis of I.

(c) If $G = \{g_1, \ldots, g_\nu\}$ is an ε-approximate border basis then the point $(c_{11}, \ldots, c_{\nu\mu})$ in $\mathbb{R}^{\nu\mu}$ given by its coefficients is **close to** the **border basis scheme**, i.e. the scheme defined by the vanishing of the coefficients of the normal remainders of the S-polynomials of the neighbor pairs of the **generic $\mathcal{O}$-border basis**.

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Thank you for your attention!