Three Differential Variants of the Buchberger-Moeller Algorithm

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4.3.2010 / Algebraic Oil Project Workshop

Outline

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Transients

- Let us depict the **state** of a production system as the entirety of
  - valve positions
  - status (test vs. production) of wells
  - working points of wells
  - etc.
- Then **transients** take place whenever any of these is changed (startup, test, optimization, shutdown).
- On a larger time scale, the depletion of a reservoir can be regarded as a transient.

Transients - Challenges

- Transients are not fully understood and in general not predictable.
- E.g., after a test phase, it can be impossible to reach the same production state as it has been previously.

Prospects

A better understanding of the dynamic behaviour of transients leading to specific procedures that guide through the testing or optimization procedures.

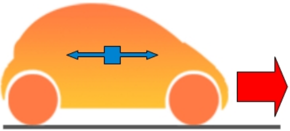
Dynamic vs. Steady-State Models

- In the first phase of the Algebraic Oil Project, among others, means to establish polynomial **steady-state models** for oil production scenarios were developed.
- Changes over time were not explicitly taken into account.
- However, it was always clear that one could (and should) do so.
- In this respect, the approach presented in the following is a **natural extension** of former results to dynamic situations.

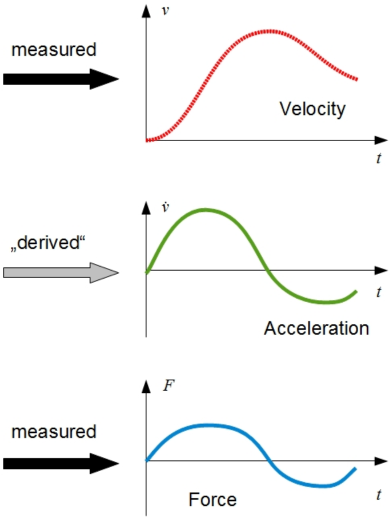
Goals

- **Extend** the AVI / ABM algorithms to the framework of differential algebra.
- **Implement** the developed methods.
- **Evaluate** and explore the methods in the context of transients scenarios coming from oil production.

Outline



$$\dot{v} \sim F$$


measured
v
Velocity
t
„derived“
$\dot{v}$
Acceleration
t
measured
F
Force
t

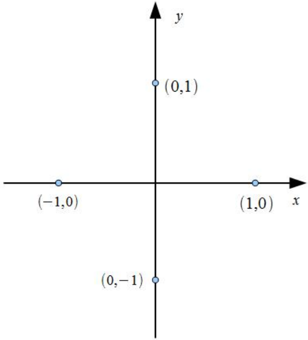
Desired features:

- **"Syntactical"** way to talk about derivatives.
- Values of derivatives do not need to be measured, but can be **deduced**.
- Relations must behave well under **further derivation**.

Differential Algebra provides the mathematical framework to this end.

Outline

From Points to Polynomials



Goal:

- **Input:** a set of points $S = \{p_1, \dots, p_s\} \subset \mathbb{Q}^n$.
- **Output:** a Groebner basis of the vanishing ideal of S.

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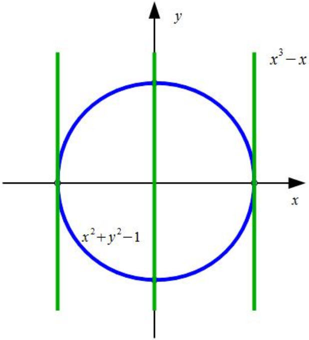

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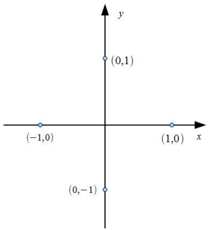
Key Ideas of the BM Algorithm



- **Order** the terms t_i.
- Consider the **evaluation vectors** $t_i(p_j)$ as columns.
- **Add** one column $(t_i(p_j))_j$ after the other,
- and find **linear dependencies** among the columns.
- "Stop when done".

	1	x	y	x^2	xy	y^2	$\dots$
$p_1 = (1,0)$							
$p_2 = (0,1)$							
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$$V(S) = \{\}$$

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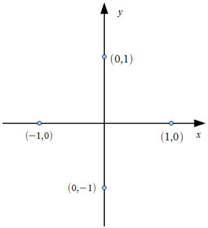

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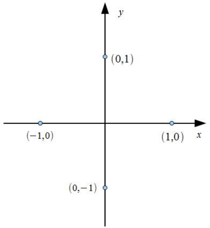

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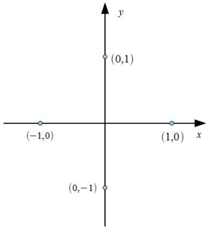

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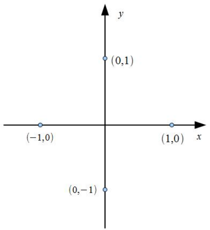

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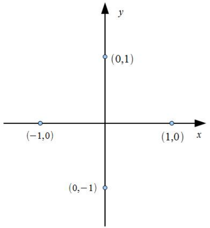

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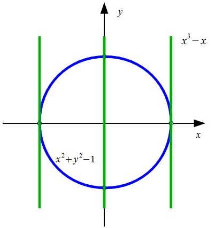

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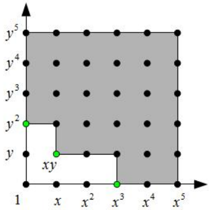

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Outline

Introducing "Differential"

- Original polynomial ring: $\mathbb{Q}[y_1, \ldots, y_n]$.
- We want to speak about **derivatives** of the y_i:

$$\dot{y}_2 = -y_1$$

- So we add them

$$D = \mathbb{Q}[y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n, \ddot{y}_1, \ldots, \ddot{y}_n, \ldots] = \mathbb{Q}\{y_1, \ldots, y_n\}$$

  to obtain the **differential polynomial ring** in the indeterminates y_i.
- Besides the **degree**, d.p.'s also have an **order**.

Introducing "Differential"

- Besides addition and multiplication, introduce a new operation, **differentiation**

$$\partial : D \longrightarrow D : f \mapsto \partial f$$

- E.g., the 1st and 2nd derivations of $f = xy \in \mathbb{Q}\{x, y\}$ are

$$\partial f = \dot{x}y + x\dot{y}$$

$$\partial^2 f = \ddot{x}y + 2\dot{x}\dot{y} + x\ddot{y}$$

- Differential ideals I are those ideals of D which are **closed** under taking derivatives:

$$f \in I \Longrightarrow \partial^k f \in I \text{ for all } k \in \mathbb{N}$$

.

Intordocing "Differential"

- Differential algebra is the algebraic treatment of differential equations.
- Differential ideals are the algebraic way to describe solutions of differential equations.

Remarks about Differential Ideals

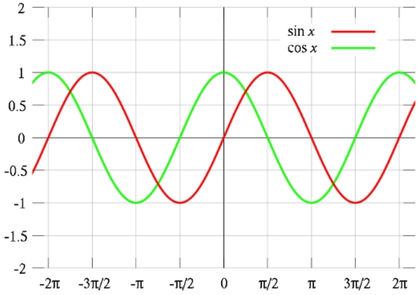
- D is not Noetherian.
- There are differential ideals that are not finitely generated.
- There are differential ideals that are not even recursive.

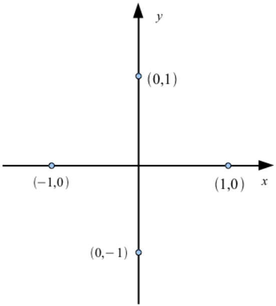
Differential Points

- In order to evaluate differential polynomials we need “values” for all differential indeterminates $y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n, \ldots$.
- We therefore require a possibly **infinite** amount of information.

Let us try to understand this at the previous example.

Differential Points





Differential Points

- Evaluate x as $\cos(p)$, $p \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$
- and y as $\sin(p)$.
- Now we extend the table downwards, taking derivatives of cos and sin into account.

	1	x	y	x^2	xy	y^2	$\dots$
φ_{p_1}	1	1	0	1	0	0	$\dots$
φ_{p_2}	1	0	1	0	0	1	$\dots$
φ_{p_3}	1	-1	0	1	0	0	$\dots$
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∂	0	$\dot{x}$	$\dot{y}$	$2\dot{x}x$	$\dot{x}y+x\dot{y}$	$2\dot{y}y$	$\dots$
φ_{p_1}	0	0	1	0	1	0	$\dots$
φ_{p_2}	0	-1	0	0	-1	0	$\dots$
φ_{p_3}	0	0	-1	0	1	0	$\dots$
φ_{p_4}	0	1	0	0	-1	0	$\dots$
∂^2	0	$\ddot{x}$	$\ddot{y}$	$\dots$			
φ_{p_1}	$\dots$						
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Differential Points

- How can we **control** the **downward growth**, i.e., how can we handle higher order derivatives?
- How can we **control** and possibly limit the **growth to the right**, i.e., higher degree differential polynomials?
- After all, is this really **useful** for practical applications?

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Overview

How can we control the downward growth, i.e., how can we handle higher order derivatives?

Variant 1

Restrict the class of possible "points".

How can we control and possibly limit the growth to the right, i.e., how can we handle higher degree differential polynomials?

Variant 2

Restrict the order of the derivatives.

After all, is this really useful for practical applications?

Variant 3

Restrict the order of the derivatives and add approximate computations to handle noisy data.

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Variant 1

Restrict the class of possible “points”.

- Idea: Assume the evaluation of the differential indeterminates is given by **recursive definitions**.
- Then it can be shown that the corresponding vanishing ideal must be finitely generated.
- If we consider only polynomials up to a certain degree, the order of the derivatives can be bounded.
- We therefore arrive at a procedure that eventually will give us a finite Groebner basis of the differential vanishing ideal.
- However, we currently do not know when to stop.

Variant 1 - The Example Continued

- For the cosine/sine example we find from the recursive definition of the evaluation, that the differential polynomials

$$\ddot{x}+x, \quad \ddot{y}+y$$

must be contained in the vanishing ideal.
- We can carry out all computations in $P = \mathbb{Q}[x, \dot{x}, y, \dot{y}]$.
- ∂ is replaced by a homogeneous $\mathbb{Q}$-linear mapping $\Delta : P \to P$.
- If we consider $P_{\leq 10}$, we compute the following set of polynomials:

$$V_{\partial}(S)_{\leq 10} = \{-y+\dot{x},\ \dot{y}+x,\ \dot{x}^2+x^2-1\}$$

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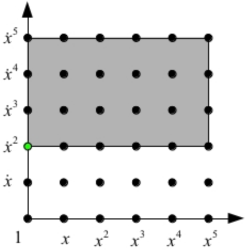
Variant 1 - Termination



- The corresponding vanishing ideals are not 0-dimensional.
- The Groebner basis must be finite.
- If we can find a degree bound is still an open question.

Variant 1 - Further Remarks

The recursive definitions of the evaluation maps is directly connected to

- particular solutions of linear differential equations with constant coefficients,
- the Taylor series of the solutions,
- certain classes of meromorphic functions.

Variant 2 - Exact but Limited Order

Restrict the order of the derivatives.

- Consider only derivatives up to order N.
- Choose an order $0 \leq M \leq N$ up to which you are interested in.
- Even though the orders are restricted, we can guarantee that with f also $\partial f, \ldots, \partial^{N-M} f$ are contained in the resulting ideal $V_{N,M}(S)$.
- Polynomials that do not survive this requirement are filtered out.

$$V_\partial(S) \cap D_M \subseteq V_{N,M}(S) \subseteq V(S) \cap D_M$$

Variant 2 - The Example Continued

	1	x	y	xy	$\dot{x}$
φ_{p_1}	1	1	0	0	0
φ_{p_2}	1	0	1	0	-1
φ_{p_3}	1	-1	0	0	0
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	1	$\dot{x}$	$\dot{y}$	$\dot{x}y + x\dot{y}$	$\ddot{x}$
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- $N = 2$, $M = 1$
- xy is filtered out.
- $\dot{x} - y$ survives.

$$V_{2,1} = \{x^4 - x^2, x^2 + y^2 - 1, \dot{x} - y, \dot{y} + x\}$$

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- $N = 2$, $M = 1$
- xy is filtered out.
- $\dot{x} - y$ survives.

$$V_{2,1} = \{x^4 - x^2, x^2 + y^2 - 1, \dot{x} - y, \dot{y} + x\}$$

Variant 3 - The Approximate Case

Restrict the order of the derivatives and add approximate computations to handle noisy data.

- Use the ideas from variant 2.
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Let us see how it works. . .

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Thank you for your attention!

Any questions?