Mathematische Software im Unterricht? Addendum

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Für die HörerInnen meines Vortrags : Die zwei großen Matrizen, die weit über den rechten Rand hinaus gehen, habe ich im Folgenden in transponierter Form dargestellt.

Was ist die Idee hinter den Groebner Basen? Linearisieren in einem ganz anderen Sinne

Wir ordnen die "Potenzprodukte" lexikographisch und nehmen die Potenzprodukte als neue Variable. Hier machen wir das einmal bis zum Grad 5 :

```
In[475]:= MR = {MacaulayRow[5]}

Out[475]= {{x[5, 5], x[5, 4], x[5, 3], x[5, 2], x[5, 1], x[5, 0], x[4, 5], x[4, 4], x[4, 3],
   x[4, 2], x[4, 1], x[4, 0], x[3, 5], x[3, 4], x[3, 3], x[3, 2], x[3, 1], x[3, 0],
   x[2, 5], x[2, 4], x[2, 3], x[2, 2], x[2, 1], x[2, 0], x[1, 5], x[1, 4], x[1, 3],
   x[1, 2], x[1, 1], x[1, 0], x[0, 5], x[0, 4], x[0, 3], x[0, 2], x[0, 1], x[0, 0]}}
```

In[476]:=
```
Transpose[MR] // MatrixForm
```

Out[476]//MatrixForm=

$$\begin{pmatrix} x[5,5] \\ x[5,4] \\ x[5,3] \\ x[5,2] \\ x[5,1] \\ x[5,0] \\ x[4,5] \\ x[4,4] \\ x[4,3] \\ x[4,2] \\ x[4,1] \\ x[4,0] \\ x[3,5] \\ x[3,4] \\ x[3,3] \\ x[3,2] \\ x[3,1] \\ x[3,0] \\ x[2,5] \\ x[2,4] \\ x[2,3] \\ x[2,2] \\ x[2,1] \\ x[2,0] \\ x[1,5] \\ x[1,4] \\ x[1,3] \\ x[1,2] \\ x[1,1] \\ x[1,0] \\ x[0,5] \\ x[0,4] \\ x[0,3] \\ x[0,2] \\ x[0,1] \\ x[0,0] \end{pmatrix}$$

Wir verwenden hier folgende Codierung : x[i, j] steht für $x^i y^j$.

p1 und p2 haben dann folgende Gestalt :

In[452]:=
```
p1M = MacaulayComposed[5, {{0, 0, 2}, {2, 0, -1}, {0, 1, 1}, {1, 1, 1}}]
p2M = MacaulayComposed[5, {{1, 1, -1}, {2, 1, 1}}]
```

Out[452]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 2}
```

Out[453]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0}
```

Zusammen in einer "Koeffizientenmatrix" :

In[473]:=
```
FM = {MacaulayRow[5], p1M, p2M}
```

Out[473]=
```
{{x[5, 5], x[5, 4], x[5, 3], x[5, 2], x[5, 1], x[5, 0], x[4, 5], x[4, 4], x[4, 3],
  x[4, 2], x[4, 1], x[4, 0], x[3, 5], x[3, 4], x[3, 3], x[3, 2], x[3, 1], x[3, 0],
  x[2, 5], x[2, 4], x[2, 3], x[2, 2], x[2, 1], x[2, 0], x[1, 5], x[1, 4], x[1, 3],
  x[1, 2], x[1, 1], x[1, 0], x[0, 5], x[0, 4], x[0, 3], x[0, 2], x[0, 1], x[0, 0]},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1,
  0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 2}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0}}
```

In[450]:=
```
Transpose[FM] // MatrixForm
```

Out[450]//MatrixForm=

x[5, 5]	0	0
x[5, 4]	0	0
x[5, 3]	0	0
x[5, 2]	0	0
x[5, 1]	0	0
x[5, 0]	0	0
x[4, 5]	0	0
x[4, 4]	0	0
x[4, 3]	0	0
x[4, 2]	0	0
x[4, 1]	0	0
x[4, 0]	0	0
x[3, 5]	0	0
x[3, 4]	0	0
x[3, 3]	0	0
x[3, 2]	0	0
x[3, 1]	0	0
x[3, 0]	0	0
x[2, 5]	0	0
x[2, 4]	0	0
x[2, 3]	0	0
x[2, 2]	0	0
x[2, 1]	0	1
x[2, 0]	-1	0
x[1, 5]	0	0
x[1, 4]	0	0
x[1, 3]	0	0
x[1, 2]	0	0
x[1, 1]	1	-1
x[1, 0]	0	0
x[0, 5]	0	0
x[0, 4]	0	0
x[0, 3]	0	0
x[0, 2]	0	0
x[0, 1]	1	0
x[0, 0]	2	0

Was ist die Idee hinter den Groebner Basen? Jetzt triangulieren wir diese Matrix :

In[470]:=
```
MCMatrix5R = RowReduce[MCMatrix5]
```

Out[470]=

{{0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, -4, 0, 0, 0, -4, -8, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, $-\frac{2}{3}$, $-\frac{4}{3}$, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, $-\frac{8}{3}$, $-\frac{22}{3}$, -4}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, $\frac{1}{3}$, $\frac{2}{3}$, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, $-\frac{2}{3}$, $-\frac{4}{3}$, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -2, 0, 0, 0, $-\frac{4}{3}$, $-\frac{8}{3}$, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, $-\frac{1}{6}$, $-\frac{1}{3}$, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, $\frac{1}{3}$, $\frac{2}{3}$, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, $-\frac{2}{3}$, $-\frac{4}{3}$, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, $-\frac{2}{3}$, $-\frac{7}{3}$, -2},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
0, 0, 0, 0, 0, 1, 0, $-\frac{31}{6}$, $-\frac{7}{3}$, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, $-\frac{1}{6}$, $-\frac{1}{3}$, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 0, 0, $\frac{1}{3}$, $\frac{2}{3}$, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, $-\frac{2}{3}$, $-\frac{4}{3}$, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 1, $\frac{5}{2}$, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

In[472]:=
```
Transpose[Prepend[MCMatrix5R, MacaulayRow[5]]] // MatrixForm
```

Out[472]//MatrixForm=

"x"[5, 5]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[5, 4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[5, 3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[5, 2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[5, 1]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[5, 0]	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[4, 5]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[4, 4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[4, 3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[4, 2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[4, 1]	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[4, 0]	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[3, 5]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[3, 4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[3, 3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[3, 2]	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
"x"[3, 1]	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
"x"[3, 0]	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
"x"[2, 5]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[2, 4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[2, 3]	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
"x"[2, 2]	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
"x"[2, 1]	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
"x"[2, 0]	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
"x"[1, 5]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[1, 4]	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
"x"[1, 3]	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
"x"[1, 2]	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
"x"[1, 1]	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
"x"[1, 0]	-4	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0
"x"[0, 5]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
"x"[0, 4]	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
"x"[0, 3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
"x"[0, 2]	-4	$-\frac{2}{3}$	$-\frac{8}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{31}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{5}{2}$	0
"x"[0, 1]	-8	$-\frac{4}{3}$	$-\frac{22}{3}$	$\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{8}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{7}{3}$	$-\frac{7}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{4}{3}$	1	0
"x"[0, 0]	0	0	-4	0	0	0	0	0	0	-2	0	0	0	0	0	0

G

$\{2\,y + 5\,y^2 + 2\,y^3,\ -4\,y + 3\,x\,y - 2\,y^2,\ -6 + 3\,x^2 - 7\,y - 2\,y^2\}$

Die Gröbner - Basis findet sich in der "Contour" der triangulierten Matrix!